Homework Assignment No. 4, due Monday, 2/13 at 5 pm

1. Let the stock price process be a geometric Brownian motion

\[ \frac{dS}{S} = r dt + \sigma d\tilde{W}(t), \quad S(0) = S_0 \]

Derive the explicit analytical expression for the risk-neutral probability of the following events:

(a) \( \{K_1 < S(T) < K_2\} \)

(b) \( \{S(T_2) > S(T_1) > S(t)\}, \quad \text{with } T_2 > T_1 > t > 0. \)

2. Assume the standard Black-Scholes model for a stock price as in Problem 1, A European call spread has payoff

\[ \Lambda(S(T)) = \begin{cases} 
0 & \text{if } S(T) \leq K \\
S(T) - K & \text{if } K < S(T) < K + D \\
D & \text{if } S(T) \geq K + D
\end{cases} \]

where \( K > 0 \) and \( D > 0 \) are given constants.

(a) Derive the formula for the option value \( V(t, S) \) and delta \( \Delta(t, S) \) at time \( t \), in terms of the spot stock price \( S(t) = S \), and time to maturity \( T - t \).

(b) Find the limits of \( V(t, S) \) as \( D \to 0 \) and \( D \to \infty \), and give a financial interpretation for your answers.