Solution Notes for Homework Assignment 6

1. (a)

\[ d \left( e^{\beta t} R(t) \right) = \beta e^{\beta t} R(t) dt + e^{\beta t} dR(t) \]
\[ = \beta e^{\beta t} R(t) dt + e^{\beta t} \left[ (\alpha - \beta R(t)) dt + \sigma dW(t) \right] \]
\[ = e^{\beta t} \left[ \alpha dt + \sigma dW(t) \right] \]

(b) Integrating from 0 to \( t \),

\[ e^{\beta t} R(t) - R(0) = \int_0^t e^{\beta s} \left[ \alpha ds + \sigma dW(s) \right] \]
\[ = \alpha \int_0^t e^{\beta s} ds + \sigma \int_0^t e^{\beta s} dW(s) \]
\[ = \frac{\alpha}{\beta} \left( e^{\beta t} - 1 \right) + \sigma \int_0^t e^{\beta s} dW(s). \]

Therefore

\[ R(t) = e^{-\beta t} R(0) + \frac{\alpha}{\beta} \left( 1 - e^{-\beta t} \right) + \sigma \int_0^t e^{-\beta(t-s)} dW(s) \]

2. Obviously \( B_1 \) is a Brownian motion. To show that for \( B_2 \), we note

\[ dB_2(t) = \rho(t) dW_1(t) + \sqrt{1 - \rho^2(t)} dW_2(t) \]

which can be viewed as a sum of two independent normal random variables, so the sum is also normal, with mean zero, and variance

\[ \rho(t)^2 dt + \left( \sqrt{1 - \rho^2(t)} \right)^2 dt = dt \]

We also need to check that \( B_2(t_2) - B_2(t_1) \) and \( B_2(s_2) - B_2(s_1) \) are independent for \( t_2 > t_1 > s_2 > s_1 \). Finally we calculate

\[ dB_1(t) dB_2(t) = dW_1(t) \cdot \left( \rho(t) dW_1(t) + \sqrt{1 - \rho^2(t)} dW_2(t) \right) \]
\[ = \rho(t) (dW_1)^2 + \sqrt{1 - \rho^2(t)} dW_1 \cdot dW_2 \]
\[ = \rho(t) dt \]
3. (a) Apply Itô’s formula to $f(x,t) = e^{-\theta x - (r + \frac{1}{2} \theta^2) t}$,

$$
\begin{align*}
  d\zeta(t) &= f_t dt + f_x dW(t) + \frac{1}{2} f_{xx} dt \\
  &= -\zeta(t) \left( r + \frac{1}{2} \theta^2 \right) dt + \zeta(t)(-\theta) dW(t) + \frac{1}{2} \zeta(t)(-\theta)^2 dt \\
  &= -\theta \zeta(t) dW(t) - r \zeta(t) dt.
\end{align*}
$$

(b) From the definition of $X(t)$, we have

$$
\begin{align*}
  dX(t) &= \Delta(t) dS(t) + r (X(t) - \Delta(t) S(t)) dt \\
  &= [\alpha \Delta(t) S(t) + r (X(t) - \Delta(t) S(t))] dt + \sigma \Delta(t) S(t) dW(t) \\
  &= \sigma \left[ \theta \Delta(t) S(t) + \frac{r}{\sigma} X(t) \right] dt + \sigma \Delta(t) S(t) dW(t)
\end{align*}
$$

Using Itô’s product rule,

$$
\begin{align*}
  d(\zeta(t) X(t)) &= \zeta(t) dX(t) + X(t) d\zeta(t) + d\zeta(t) \cdot dX(t).
\end{align*}
$$

We collect the coefficients for $dt$ terms:

$$
\zeta(t) \sigma \left[ \theta \Delta(t) S(t) + \frac{r}{\sigma} X(t) \right] - r X(t) \zeta(t) - \sigma \theta \zeta(t) \Delta(t) S(t) = 0.
$$

So $d(\zeta(t) X(t))$ only has $dW(t)$ terms, which means that $\zeta X$ is a martingale.

(c) We have demonstrated that by choosing $\Delta(t)$ properly, we can replicate the payoff of a derivative $V(T) = F(S(T))$ with a portfolio $X(t)$ such that

$$
X(T) = V(T),
$$

and based on no-arbitrage principle, we must have $X(t) = V(t)$ for all $0 \leq t \leq T$. In particular, if $\zeta(t) X(t)$ is a martingale, we have for $t = 0$

$$
X(0) = \zeta(0) X(0) = E[\zeta(T) X(T)] = E[\zeta(T) V(T)].
$$

This formula expresses the initial value of the portfolio, therefore the derivative price at $t = 0$, as an expected value of a random variable under the original probability measure. Note that this differs from the risk-neutral pricing formula in that not only the discount factor is included in $\zeta(t)$, but also a factor that resembles the R-N derivative we saw before.