Solutions to Homework Assignment 3

1. We should just recognize that \( W(t + \Delta t) - W(t) \) is a normal random variable with mean 0 and variance \( \Delta t \), and the definition of the kurtosis.

2. (a) It is straightforward to see
\[
\frac{S(t_{j+1})}{S(t_j)} = \exp \left( (r - \frac{1}{2}\sigma^2) \Delta t + \sigma \Delta W_j \right),
\]
so
\[
\sum_{j=0}^{n-1} \left( \log \frac{S(t_{j+1})}{S(t_j)} \right)^2 = \sum_{j=0}^{n-1} \left( (r - \frac{1}{2}\sigma^2) \Delta t + \sigma \Delta W_j \right)^2
\]
\[
= \sigma^2 \sum_{j=0}^{n-1} (\Delta W_j)^2 + (r - \frac{1}{2}\sigma^2)^2 \sum_{j=0}^{n-1} (\Delta t)^2 + 2\sigma (r - \frac{1}{2}\sigma^2) \sum_{j=0}^{n-1} (\Delta W_j) \cdot \Delta t.
\]

(b) As \( \Delta t \to 0 \),
\[
\sum_{j=0}^{n-1} (\Delta W_j)^2 \to \sum_{j=0}^{n-1} \Delta t = T, \]
\[
\sum_{j=0}^{n-1} (\Delta t)^2 = \Delta t \sum_{j=0}^{n-1} \Delta t = T \Delta t \to 0, \]
\[
\sum_{j=0}^{n-1} \Delta W_j \cdot \Delta t = \frac{T}{n} \sum_{j=0}^{n-1} \Delta W_j \to 0.
\]
The last part is based on the law of large numbers. After adding up these three parts, we have the limit \( \sigma^2 T \).

(c)
\[
\sum_{j=0}^{n-1} Y_j = (r - \frac{1}{2}\sigma^2)T + \sigma W(T) = \log \frac{S(T)}{S(0)},
\]
so the term that should be subtracted from the sum is
\[
\frac{1}{n} \left( \log \frac{S(T)}{S(0)} \right)^2,
\]
which can be easily implemented in any estimate. To answer the question when it is justified to leave out this correction term, we note

\[ \frac{1}{n} \left( \sum_{j=0}^{n-1} Y_j \right)^2 = \frac{1}{n} \left( \alpha^2 T^2 + 2\alpha\sigma T^{3/2} Z + \sigma^2 T Z^2 \right) \]

where \( Z \) is a standard normal random variable and \( \alpha = r - \sigma^2/2 \). The mean of the sum is

\[ \frac{1}{n} \left( \alpha^2 T^2 + \sigma^2 T \right) , \]

and the variance of the sum is

\[ \frac{1}{n^2} \left( 4\alpha^2 \sigma^2 T^3 + 2\sigma^4 T^2 \right) . \]

We claim that as \( n \) becomes very large, both the mean and the variance of this correction term approach zero, therefore it is justified to drop that correction term when \( n \) is very large, or \( \Delta t \) approaches zero.

(d) If \( r \) and \( \sigma \) are both time dependent, we need to adjust the model to

\[ S(t) = S(0) \exp \left( \int_0^t \left( r(u) - \frac{1}{2} \sigma^2(u) \right) du + \int_0^t \sigma(u) dW(u) \right) . \]

The sum in question becomes

\[ \sum_{j=0}^{n-1} \left( \log \frac{S(t_{j+1})}{S(t_j)} \right)^2 \]

\[ = \sum_{j=0}^{n-1} \left( (r_j - \frac{1}{2} \sigma_j^2) \Delta t + \sigma_j \Delta W_j \right)^2 \]

\[ = \sum_{j=0}^{n-1} \sigma_j^2 (\Delta W_j)^2 + \sum_{j=0}^{n-1} (r_j - \frac{1}{2} \sigma_j^2)^2 (\Delta t)^2 + 2 \sum_{j=0}^{n-1} \sigma_j (r_j - \frac{1}{2} \sigma_j^2) (\Delta W_j) \cdot \Delta t \]

The first term will go to

\[ \int_0^T \sigma^2(t) dt \]

and the other two will also approach zero, as long as \( r(t) \) and \( \sigma(t) \) satisfy some boundedness conditions.

3. Let \( f(t, x) = \log x \),

\[ d \log S = \frac{1}{S} dS + \frac{1}{2} \left( \frac{1}{S^2} \right) (dS)^2 \]

\[ = \alpha dt + \sigma dW - \frac{1}{2} \sigma^2 dt \]

\[ = \left( \alpha - \frac{1}{2} \sigma^2 \right) dt + \sigma dW \]

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