Solutions for Assignment No. 6

Problem 2.10

(i) First we calculate the conditional expectation

\[ \tilde{E}_n[X_{n+1}] = \Delta_n S_n \tilde{E}_n[Y_{n+1}] + (1 + r)(X_n - \Delta_n S_n) \]

\[ = \Delta_n S_n (\tilde{p}u + \tilde{q}d) + (1 + r)(X_n - \Delta_n S_n) \]

\[ = (1 + r)X_n + \Delta_n S_n (\tilde{p}u + \tilde{q}d - 1 - r) \]

\[ = (1 + r)X_n. \]

This shows that \( X_n / (1 + r)^n \) is a martingale under the risk-neutral measure.

(ii) If we choose \( \Delta_n \) to be

\[ \Delta_n = \frac{V_{n+1}(H) - V_{n+1}(T)}{S_n(u - d)}, \]

then we will have \( X_{n+1}(H) = V_{n+1}(H) \) and \( X_{n+1}(T) = V_{n+1}(T) \) (verify!). Again, we have replicated the payoff of the derivative, so we can claim \( V_n = X_n \) for all coin tosses at any time \( n \). In particular, since the discounted portfolio price \( X_n / (1 + r)^n \) is a martingale under the risk-neutral measure, as shown in part (i), we have the pricing formula

\[ V_n = \frac{1}{1 + r} \tilde{E}_n[V_{n+1}] . \]

(iii)

\[ \tilde{E}_n[S_{n+1}] = \tilde{E}_n[(1 - A_{n+1})Y_{n+1}S_n] \]

\[ = S_n \tilde{E}_n[Y_{n+1}] - S_n \tilde{E}_n[A_{n+1}Y_{n+1}] \]

\[ = S_n (\tilde{p}u + \tilde{q}d) - S_n \tilde{E}_n[A_{n+1}Y_{n+1}] \]

\[ = (1 + r)S_n - S_n \tilde{E}_n[A_{n+1}Y_{n+1}] . \]

Since \( \tilde{E}_n[A_{n+1}Y_{n+1}] \) in general is not zero, the discounted \( S_n \) is not a martingale under the risk-neutral measure.

However, if \( A_{n+1} = \alpha \) is a constant,

\[ \tilde{E}_n[A_{n+1}Y_{n+1}] = \alpha \tilde{E}_n[Y_{n+1}] = \alpha (1 + r) . \]

Therefore,

\[ \tilde{E}_n[S_{n+1}] = (1 + r)S_n - \alpha (1 + r)S_n = (1 - \alpha)(1 + r)S_n , \]

which implies that \( S_n / ((1 - \alpha)^n(1 + r)^n) \) is a martingale under the risk-neutral measure.
Problem 2.11

(i) The payoff of the forward contract that requires you to buy one share of the stock at time $N$ for $K$ dollars is $S_N - K$, and the payoff of the put with strike $K$ at expiration is $(K - S_N)^+$. The combination is therefore

$$S_N - K + (K - S_N)^+ = \begin{cases} 0, & S_N \leq K \\ S_N - K, & S_N > K \end{cases} = (S_N - K)^+.$$

This shows $F_N + P_N = C_N$.

(ii) All three securities are derivatives of the same stock so the general pricing formula applies:

$$C_n = \tilde{E}_n \left[ \frac{C_N}{(1 + r)^{N-n}} \right],$$

$$P_n = \tilde{E}_n \left[ \frac{P_N}{(1 + r)^{N-n}} \right],$$

and

$$F_n = \tilde{E}_n \left[ \frac{F_N}{(1 + r)^{N-n}} \right],$$

for $0 \leq n < N$, using part (i) and linearity of conditional expectations, we have $F_n + P_n = C_n$.

(iii) We start with the payoff at time $N$ for the forward contract

$$F_N = S_N - K,$$

and divide both sides by $(1 + r)^N$, then take the expectation under the risk-neutral probability measure,

$$F_0 = \tilde{E} \left[ \frac{F_N}{(1 + r)^N} \right] = \tilde{E} \left[ \frac{S_N}{(1 + r)^N} \right] - \frac{K}{(1 + r)^N} = S_0 - \frac{K}{(1 + r)^N}.$$

(iv) Since the stock price at time zero is $S_0$, to buy one share you would need to borrow an extra amount $S_0 - F_0 = K/(1 + r)^N$ (if that amount is negative you just deposit it in a bank rather than borrow). At time $N$, your stock is worth $S_N$ and your outstanding debt is $(S_0 - F_0)(1 + r)^N = K$ so the total portfolio value is $S_N - K$.

(v) If we set $F_0 = 0$, we must choose $K = (1 + r)^N S_0$. With this particular choice of $K$, $C_0 = P_0$ based on part (ii).

(vi) Suppose we choose a value $K = (1 + r)^N S_0$ such that $F_0 = 0$, there is no reason for the price of the forward contract $F_n$ to stay zero, therefore according to part (ii) we will not have $C_n = P_n$ for $n > 0$ in general.