Midterm Project, Math 5760, due Oct. 24

1. Implement an N-period binomial tree model for a stock process $S_j, j = 0, 1, \ldots, N$, where S_j represents the stock price at time $t_j = j\Delta t$, $\Delta t = T/N$ for a fixed expiration T. The stock price is assumed to have a constant volatility σ, and the risk-free interest rate $r > 0$ is also fixed. According to the model with μ taken to be zero, the stock price process can be summarized as

$$\log S_j = \log S_{j-1} + \sigma \Delta t Z_j$$

   where

$$Z_j = \begin{cases} 1 & \text{with probability } p \\ -1 & \text{with probability } 1 - p \end{cases}$$

   and the risk-neutral probability

$$p = \frac{1}{2}\left(1 + \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)\sqrt{\Delta t}\right).$$

   Program this model on any of the following platforms: Matlab, Maple, or C/C++. The input parameters should include: current stock price S_0, time to expiration T, number of time periods N, volatility σ (in percentage), and risk-free interest rate r. The program should be capable of accepting any payoff function in the form $F(S)$, for example $F(S) = \max(S - K, 0)$ for a call option, and $F(S) = \max(K - S, 0)$ for a put option.

   Note: it is also possible to implement the model on an Excel spreadsheet, though the use of different N values can be inconvenient.

2. Use the binomial model and the Black-Scholes formula to price European call options on a stock (with no dividend) expiring in 3 months ($T = 0.25$), with strikes at \$90, \$95, \$100, \$105, and \$110. The current stock price is \$100. Use volatility levels at 10% 20% and 30%. The risk-free interest rate is assumed to be 1%. Compute the differences between the binomial model and the Black-Scholes formula for all different strikes and volatilities with number of time periods $N = 10, 50$ and 100.

3. The close price of Apple stock on Friday was \$492.81. The December options (expiring at close Saturday, Dec. 21, 2013) with strikes \$480, \$485, \$490, \$495, \$500 were quoted with prices \$29.7, \$27.05, \$24.6, \$22.15, and \$20.10 at Friday's close. Use any numerical solver to invert the Black-Scholes formula to obtain the implied volatilities for these options, and plot them as a function of strike to demonstrate any smile or skew phenomenon. You can assume zero interest rate to make calculations simpler.

4. Extend the binomial model to price an American put option with expiration $T = 0.25$, strike $K = \$95$, on a non-dividend-paying stock, using $N = 60$. What is the price difference between the American put and the corresponding European put? If the option holder is allowed to exercise after every 10 steps (at $n = 10, 20, 30, 40, 50$ and 60), what is the value of this option? (it is called a Bermudan put.)

5. Suppose the current stock price is \$25, and volatility $\sigma = 25\%$, use the binomial model to price the following exotic options:

   (a) a digital option: if the stock price at $T = 1$ is greater than or equal to \$30, the option holder receives \$1, otherwise the holder receives nothing.

   (b) a call on a call option: with this compound option, at time $T_1 = 0.25$ the holder has the right to pay $K_1 = \$1$ to receive a call option with a strike $K_2 = \$25$ that expires at $T_2 = 0.5$. The compound option in question is to be priced at time zero.

   Hint: the payoff of the compound option at T_1 is $\max(C_1 - 1, 0)$, where C_1 is the price at T_1 of a call option with payoff $\max(S_2 - 25, 0)$ at T_2, and S_2 is the stock price at T_2.