Note for Problem 3

This is an analysis of the binomial model for the early exercise problem. First we show for the non dividend case that you should never exercise earlier than the expiration. Let's look at the step that is right before the expiration, when we are at a node for $t = (N-1)\Delta t$ and the stock price is S_{N-1}. Because of the martingale property,

$$S_{N-1} = e^{-r\Delta t}\mathrm{E}[S_N] = e^{-r\Delta t}\left[pS_{N-1}u + (1-p)S_{N-1}d\right]$$

we have

$$S_{N-1} - K < S_{N-1} - Ke^{-r\Delta t} = e^{-r\Delta t}\left[p(S_{N-1}u - K) + (1-p)(S_{N-1}d - K)\right]$$

On the other hand, since $X < \max(X, 0)$,

$$S_{N-1} - K < e^{-r\Delta t}\left[p(\max(S_{N-1}u - K, 0) + (1-p)\max(S_{N-1}d - K, 0)\right] = e^{-r\Delta t}\mathrm{E}[\max(S_N - K, 0)]$$

The right-hand-side is always nonnegative, and If the left-hand-side is negative, the early exercise question does not arise. If the left- is positive, it is less than the value of continuation so the option is not exercised.

Now consider a time t_j when $j < N-1$, we still have

$$\begin{aligned} S_j - K &< S_j - Ke^{-r\Delta t} \\ &= e^{-r\Delta t}\mathrm{E}_j[S_{j+1} - K] \\ &\le e^{-r\Delta t}\mathrm{E}_j[\max(S_{j+1} - K, 0)] \\ &\le e^{-r\Delta t}\mathrm{E}_j[C_{j+1}] \end{aligned}$$

for $j = N-1, N-2, \ldots, 0$. The last inequality is based on the fact that at $j+1$ we do not exercise the option because the value of immediate exercise is less than the value of continuation. This proves that the option should never be exercised before expiration. Notice the crucial factor is the fact that $r > 0$ so

$$S - K < S - Ke^{-r\Delta t}$$

Now back to our 4-step model, at $j = 2$ and 3, we can use the conclusion from above as there is no more dividend payments in the horizon. The only time we should consider the early exercise situation is at $j = 1$. Suppose we are at a node at time $j = 1 (t = 0.25)$ and the stock price is S_1, after one step the stock price should be $S_1 u$ and $S_1 d$ respectively according to the move, then there is a price cut due to the dividend payment at $j = 2$ so the stock price should be $S_1 u(1-\delta)$ and $S_1 d(1-\delta)$ respectively. Our job is to compare the immediate exercise at $j = 1$ with payoff $V_e = S_1 - K$ (assuming it is positive because otherwise we would never need to consider early exercise) with the value of continuation V_c.

$$\begin{aligned} V_c &= e^{-r\Delta t}\mathrm{E}_1[C_2] \\ &\ge e^{-r\Delta t}\mathrm{E}_1[\max(S_2(1-\delta) - K, 0)] \\ &\ge e^{-r\Delta t}\mathrm{E}_1[S_2(1-\delta) - K] \\ &= (1-\delta)S_1 - Ke^{-r\Delta t} \end{aligned}$$

The final step is to show

$$(1-\delta)S_1 - Ke^{-r\Delta t} \ge S_1 - K$$

when $S_1 < K$ and $\delta < r\Delta t$. When that happens, the option is not exercised.