Homework Assignment No.9, Due Tuesday, Nov 19 at 5 pm

1. Consider the one-step trinomial tree, where the stock price after one time period will be one of the following: S_0u, S_0 or S_0d. Here S_0 is the stock price at the beginning of the time period, $u > 1$ and $d < 1$ are given, and $r > 0$ is the annualized risk-free interest rate.

   (a) Derive the conditions for the risk-neutral probability measure, that is the probabilities p_1, p_2 and p_3 of those three outcomes respectively.

   (b) Suppose in addition to the stock, a call option on the stock is also traded, with a payoff at the end of the time period to be $S_0(u-1)$ if the stock price reaches S_0u, and zero otherwise. Find a risk-neutral probability measure that leads to no-arbitrage in this model.

2. Let W_t be a Brownian motion. Determine for each of the following events if it is in the filtration of $\mathcal{F}_s$ where $r < s < t$. Explain your answers.

   (a) $W_s > 1$.

   (b) $W_t > 2W_s$.

   (c) $W_s > W_{s-2}$.

   (d) $W_r < 0$.

3. For the four-step binomial model with a dividend paid at $t = 0.5$ in the last assignment, prove the following results:

   (a) If $\delta \leq r$, the American call has the same value as the corresponding European call, that is, you should never exercise the option earlier than the expiration time.

   (b) If $\delta > r$ and $S_0 > K$, the American call has a higher value than the corresponding European call.

4. Consider a derivative with the payoff function $F(S_T) = \log S_T^2$ and expiration T, and assume the Black-Scholes model for the stock price S_t. Assume that the risk-free interest rate is r and the volatility of the stock is σ. Price this derivative by evaluating the expectation of the discounted payoff under the risk-neutral probability measure.

5. Suppose a stock S_t follows a geometric Brownian motion with a time-dependent volatility. It is observed that the current stock price $S_0 = 100$ and $r = 1\%$, and the 3-month and 6-month European call options with strike 100 have implied volatilities 20% and 25% respectively. Find a piecewise constant volatility function that is consistent with the implied volatilities.