Homework Assignment No.7, Due Tuesday, Nov 5 at 5 pm

1. Use the Matlab code `bm_path.m` (a link can be found on the class web page) to study the procedure to approximate Brownian paths by random walks. Assume $M = 100$ paths and $N = 100$ steps, compute the sample variances for X_t at $t = 0.5$ and $t = 1$. What should we expect from these two variances based on our understanding of $\mathbb{V}ar(X_t)$ as a function of t? Compute the sample variances for Y_t at $t = 0.5$ and $t = 1$ and comment on the relation between these two variances.

2. Use the same code with $M = 100$ and $N = 100$, $\mu = 0.05$ and $\sigma = 0.25$, compute the sample means of Y_T and $\max(Y_T - 1, 0)$ with $T = 1$. Which exact results should we use to compare them with? Now double the number of paths, keeping everything else the same, compute the errors and comment on the convergence of the approximation as we use more and more paths.

3. Suppose X_t and Y_t are both geometric Brownian motions following

$$\frac{dX_t}{X_t} = \mu_X\, dt + \sigma_X\, dW_t, \qquad \frac{dY_t}{Y_t} = \mu_Y\, dt + \sigma_Y\, dW_t$$

   for some given constants $\mu_X, \mu_Y, \sigma_X > 0$, and $\sigma_Y > 0$. Use Itô's lemma to derive the process for X_t/Y_t. Determine if this process is also a geometric Brownian motion.

4. If we assume that the stock price follows Brownian motion

$$dS_t = \mu\, dt + \sigma\, dW_t,$$

   rather than the geometric Brownian motion as in the Black-Scholes model, we can still follow the idea of constructing a riskless portfolio consisting of the stock and the option, and set its return to be the riskless interest rate based on the no-arbitrage principle. Construct such a portfolio and derive the PDE for the option price for this model, which is an analogue of the Black-Scholes equation.

Suggested problems in the text to study but not to turn in: 5.1, 5.3, 5.5.