Homework Assignment No.5, Due Thursday, Oct. 10 at 5 pm

This set of exercises concern the 4-step binomial tree we worked out in class. A spreadsheet named `4step_model.xlsx` is attached for your convenience. Notice that in this model, we assume $S_0 = 100, T = 1, r = 2\%, \sigma = 0.2 = 20\%$, and we used backward induction to price a European call with strike 100.

1. Use the binomial tree provided on the spreadsheet to price a call option with expiration $T = 1$ and strike $K = 110$;

2. Use the same tree to price an American call with $T = 1$ and $K = 100$, and provide with the optimal exercise policy that lists a detailed instruction as when to early exercise and when not to. Is there any difference between the prices of American call and European call with the same parameters?

3. Use the same tree to price an American put with $T = 1$ and $K = 110$.

4. At any node of the tree, we can calculate the hedge ratio (delta) of the derivative under consideration. For example, if you look at cell E35, it is the delta of the European call at the beginning before any price moves. It is based on

$$\delta = \frac{C_+ - C_-}{S_+ - S_-}$$

for this particular node, where S_+ and C_+ are the stock price and option price at the next time step *if* the stock move was upward, and likewise S_- and C_- are the stock price and option price *if* the stock move was downward. In this case, $S_+ = 110.52, C_+ = 14.24, S_- = 90.48, C_- = 2.73$, so we have $\delta = 0.574$ for this node. Similarly, if the first stock move is upward, then the delta changes to 0.797, because now $S_+ = 122.1, C_+ = 23.13, S_- = 100, C_- = 5.47$. The deltas can be used to provide a detailed instruction as how to hedge your position if you sold such a call. For example, you are trading for a financial firm and you just sold such a call and received the market price \$8.44. Immediately you need to hedge this position so that you will have the exact to pay to the option holder at expiration. Here are the steps you can take based on the stock price moves. We check for one particular stock price path: $S_0 = 100 \to 110.52 \to 122.14 \to 134.98 \to 149.18$ (shown on row 42):

   (a) At $t = 0$, the delta is 0.574, which is the number of stock shares you need in your portfolio. The stock price is 100 so you need \$57.4 to purchase the stock. However, your proceed from the sale of the option is only \$8.43, so you need to take a loan with amount \$49.01 (cells E43-45).

   (b) Next the stock moves to 110.52 so your portfolio worth is now $0.574 \times 110.52 - 49.01 \times e^{0.02 \times 0.25} \approx 14.23$, but the new delta is 0.797 which requires you to buy more stock. You need to take 0.797×110.52 out of your portfolio to realize this position and your loan position is increased to 73.88.

   (c) Next when the stock price is 122.14, your portfolio is worth 23.13. As you can see, by following this delta hedging strategy repeatedly for later steps, you will have a varying portfolio with values (E45-I45) tracking the values of the option (E29-I29), for this particular path of stock prices.

   (d) If we can show that the option prices are exactly matched by your strategy for a portfolio for *every stock path*, then the option is *replicated*, which justifies the initial price of \$8.438.

   Show for the following stock price paths that the final portfolio value exactly matches the option payoff value:

(a)

$$100 \to 90.48 \to 100 \to 110.52 \to 100$$

(b)

$$100 \to 90.48 \to 81.87 \to 90.48 \to 100$$