Final Practice Problems, Math 5760/6890, Fall 2013

Instructions: This is an open book but closed note exam. Calculators should be used but wireless devices including laptops are not allowed. You need to show all the details of your work to receive full credit.

1. Consider a one-step model for a stock price in which the end price $S_1 = S_0(1+X)$, where S_0 is the price at the beginning and X is a random variable with mean α and variance σ^2. Assume that the interest rate over this unit period is R and an investor can borrow any amount at this rate, the investor can form a portfolio that consists of a loan and a number of shares of the stock, such that the expected return of the portfolio is $\mu > \alpha$. What is the variance of the return over this unit period? What is the volatility of the portfolio, assuming that it is quoted with respect to this time unit?

2. In each of the following situations, determine if there is any arbitrage opportunity. If there is, construct such an arbitrage portfolio. If not, explain why.

   (a) The stock is currently traded at $50 per share, and the European call and put with the same strike $55, and expiration in 3 months are quoted at $1 and $6 respectively. The risk-free interest rate is 1%.

   (b) Same as above, except that the risk-free rate is 10%.

   (c) The stock is traded at $50 per share, and the European calls with strikes $50 and $45, and expiration in 3 months, are quoted at $1 and $6. The riskfree interest rate is zero.

3. For each of the following pairs of prices of non-dividend paying stock S and 1-year risk less zero-coupon bond Z with principal $1,

   - $S = 100, Z = 1,$
   - $S = 90, Z = 1,$
   - $S = 110, Z = 0.9,$
   - $S = 110, Z = 1,$

   find optimal rational bounds on the following 1-year contracts

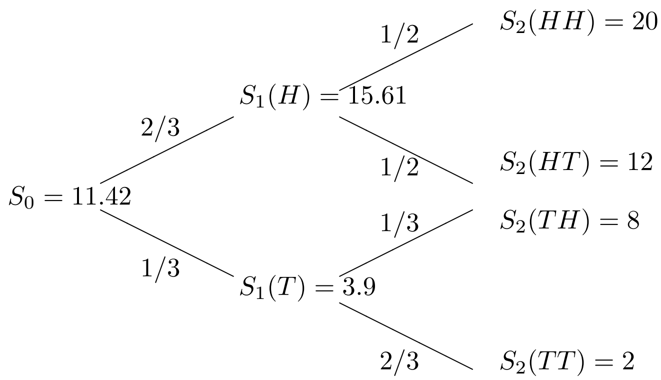
   - a digital put struck at 100;
   - a portfolio of 0.5 digital call struck at 100 and one call option struck at 110.

4. In our prototype one-step binomial model for a stock price, we assume the time period is one half year, and use $u = 2$, $d = 1/2$ for the up and down moves respectively, and a money market rate $r = 5\%$ (annualized) for the interest rate over this time period. What are the risk-neutral probabilities $\tilde{p}$ and $\tilde{q}$? What is the annualized

volatility (in percentage) implied from this model? Suppose the current stock price is \$10, and you wrote a put option with strike \$8, how much is the worth of this put option based on the no-arbitrage principle? How do you hedge this put option using the stock and a money market account?

5. In a two-step binomial model with $\Delta t = 0.5$ with $u = 1.5, d = 0.5$, and $r = 0$, calculate the risk-neutral probabilities and the distribution of the stock price at the end of two steps ($T = 1$). Suppose the current stock price is \$50, compute the European call and put prices with strike \$50 and expiration $T = 1$. Suppose you sold that put, how many shares of the stock should you buy or sell at $t = 0$, and how many should you hold at $t = 0.5$ if the first move is a down move?

6. In a two-step binomial model with $\Delta t = 0.5$, the stock prices and risk-neutral probabilities are displayed in the following, and the annualized risk-less interest rate is $r = 5\%$. Price an European call option on the stock with strike $K = \$10$ that expires at $T = 1$.



7. If
$$\frac{dS_t}{S_t} = r\,dt + \sigma\,dW_t,$$
   (a) Derive the process for S_t^2.
   (b) Does S_t^2 follow a geometric Brownian motion? If the answer is yes, write down the Black-Scholes model price as an integral for a derivative with payoff $(S_T^2 - K)^+$.

8. Price a European derivative with payoff $\log S_T$ in the Black-Scholes model. What parameters should you need in order to price it in this model?

9. Use the Black-Scholes formula to price the put in Problem 3, using a volatility of 75%. The put price in the formula is given by

$$P = Ke^{-rT}N(-d_2) - S_0N(-d_1),$$

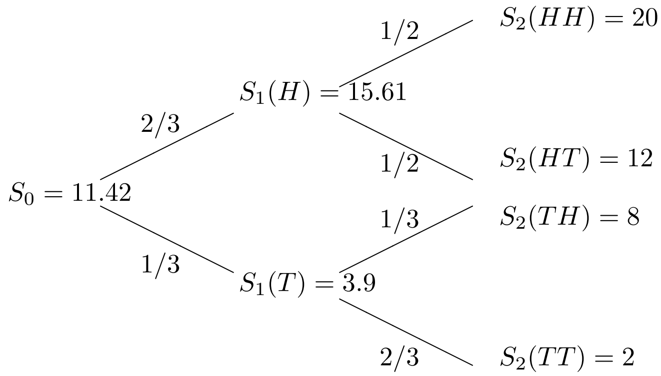
where

$$d_1 = \frac{\log(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}.$$

What is the implied volatility for the put price from Problem 3? You can use any trial-and-error approach to get an approximation within a price error of 10 cents.

10. A stock S_t follows a geometric Brownian motion with time-dependent volatility. We have $S_0 = 100$, $r = 0\%$. Put options struck at 100 with expirations 0.3, 0.6 and 1 have implied volatilities $10\%, 15\%$ and 20%. Find a piecewise constant volatility function that is consistent with these implied volatilities. Is this piecewise function the only way to fit the implied volatilities? Explain.

11. Price the American put option with strike $K = \$10$ in the same model as in Problem 3. For each node in the model, determine whether the option should be exercised or not.



12. An up-and-in call is a barrier option where the call option comes into existence only if the barrier is reached before the expiration. In the following, we have a three-period model with $\Delta t = 1/12$, and the European call option with expiration at $T = 1/4$ and strike $K = \$6$ is valid only if the stock price reaches level 16 before T. In particular, the payoff of the barrier option at T is

$$\max(S_3 - 6, 0) \cdot \mathbb{I}_{\{\tau<3\}}$$

where the stopping time $\tau = \min\{n : S_n = 16\}$. For simplicity, we assume the interest rate is zero and all probabilities of up and down moves are 1/2. Use our general binomial pricing model to price this barrier option at time zero. Should this option always be priced lower than the corresponding standard European call with the same strike and expiration? Why?