Homework Problem Notes: Chapter 2

2.15 One requirement for a replicating strategy is that whatever you specify in the strategy, a robot can execute for you, and the cashflows from your portfolio will exactly match that of the original contract, in *any state*.

To replicate the contract in question, we specify the following:

- Buy one share of the stock at t_0 (now), costing S_{t_0}, and sell it at t_2, receiving S_{t_2};
- Short sell $e^{-r(t_2-t_1)}$ shares of the stock at t_0, receiving $e^{-r(t_2-t_1)}S_{t_0}$, borrow $e^{-r(t_2-t_1)}S_{t_1}$ to buy back at t_1 and close the short position, then finally pay the loan at t_2, which becomes S_{t_1} at t_2.

The above portfolio (a long position of one share and a short position of $e^{-r(t_2-t_1)}$ shares) will generate a cashflow of $S_{t_2} - S_{t_1}$ at time t_2, and the price is

$$(1 - e^{-r(t_2-t_1)})S_{t_0}$$

2.16 This is a good example of using linear programming to obtain optimal price bounds. Suppose we buy α share of the stock, and β units of the bond, the cost today is $P = \alpha S_0 + \beta Z$, and it will turn to $\alpha S_T + \beta$ at the later time T, which is used to bound the particular derivative payoff. The function P to be minimized is linear, but subject to several constraints. Linear programming allows you check only on these (α, β) vertices to locate the maximum or the minimum, which means that we will only need to check a few combinations of α and β and choose the optimal one from these combinations.

(i) a digital call struck at 100

- Upper bounds: we will only need to compare the two bounds:

$$P_1 = Z,\ (\alpha = 0, \beta = 1);\quad P_2 = \frac{1}{100}S,\ (\alpha = \frac{1}{100}, \beta = 0)$$

  and use the smaller value to obtain the optimal upper bound.
- Lower bounds: the only bound that touches the bottom part of the payoff is 0.

(ii) a digital put struck at 100

- Upper bounds: only one possible bound that touches the payoff from above: $P = Z$.

- Lower bounds: two possible bounds that touch the bottom of payoff:

$$P_1 = 0, \quad P_2 = Z - \frac{1}{100}S,$$

  and we just need to choose the larger value to obtain the optimal lower bound.

(iii) a portfolio of 0.5 digital calls struck at 90 and one call option struck at 110

- Upper bounds: only one possible bound that is above the payoff function for all S value: $P = S$.
- Lower bounds: three possible bounds that touch the bottom of payoff:

$$P_1 = 0, \quad P_2 = \frac{1}{40}(S - 90Z), \quad P_3 = S - 109.5Z$$

  Here we show how the results are obtained:

  - $S = 100, Z = 1$: when we plug in to these three functions we get $0,\ 0.25,\ -9.5$ and the maximum is 0.25;
  - $S = 90, Z = 1$: we get $0,\ 0,\ -19.5$ and the maximum is 0;
  - $S = 100, Z = 0.9$: we get $0,\ 19/40,\ 1.45$ and the maximum is 1.45;
  - $S = 110, Z = 1$: we get $0,\ 0.5,\ 0.5$ and the maximum is 0.5.

(iv) a portfolio of 0.5 digital calls struck at 90 and one digital call option struck at 110

- Upper bounds: two bounds that touch the point $S = 110, P = 1.5$:

$$P_1 = 1.5Z, \quad P_2 = \frac{1.5}{110}S$$

- Lower bounds: only $P = 0$