Final Practice Problem Solutions

3. The risk-neutral probability

$$\tilde{p} = \frac{e^{rT} - d}{u - d} \approx 0.35, \quad \tilde{q} \approx 0.65$$

Annualized volatility σ solved from

$$\sigma^2 T = Var[S_T/S_0] = \tilde{p}u^2 + \tilde{q}d^2 - (\tilde{p}u + \tilde{q}d)^2 \approx 0.512$$

so $\sigma \approx 1.012$ or 101.2%.

Value of the put today

$$P = e^{-rT}(0.35 \times 0 + 0.65 \times 3) \approx 1.9$$

To hedge this put (assuming that you sold one), we calculate the hedge ratio to be $\Delta = \frac{0-3}{20-5} = -1/5$, so we do the following at time 0

- short 1/5 share of the stock, receive $\$10 \times 1/5 = \$2$;
- deposit $2 + 1.9 = 3.9$ into a risk-free bank account with interest rate $r = 5\%$.

You should double check the effect of the hedge in both outcomes. For example, if S moves up to 20, the portfolio leads to

- pay 0 for the put;
- pay $20/5 = 4$ to buy back the stock;
- receive $3.9e^{rT} \approx 4$ from the bank deposite.

and the net effect is zero.

A similar verification should be done for the other outcome ($S = 5$).

4.

$$C = e^{-r\Delta t}\left(\frac{2}{3} \times 6e^{-r\Delta t} + 0\right) \approx 3.8$$

5.

$$\frac{dS_t^2}{S_t^2} = (2r + \sigma^2)\,dt + 2\sigma\,dW_t$$

The solution for S_t^2 can be obtained by squaring the solution for S_t

$$S_t^2 = S_0^2 \exp\left((2r - \sigma^2)t + 2\sigma W_t\right)$$

To find the Black-Scholes price, we compare

$$C(S,K,t,T;\sigma,r) = e^{-rT}E[(S_T-K)^+]$$

with

$$e^{-rT}E[(S_T^2-K)^+]$$

It says that if S_T is lognormal such that

$$\log S_T \sim N\left(\log S_0 + \left(r - \frac{1}{2}\sigma^2\right)T, \sigma^2 T\right)$$

then

$$E[(S_T-K)^+] = e^{rT}C(S,K,t,T;\sigma,r)$$

Now S_T^2 is lognormal such that

$$\log S_T^2 \sim N\left(\log S_0^2 + \left(2r-\sigma^2\right)T, 4\sigma^2 T\right)$$

or

$$\log S_T^2 \sim N\left(\log S_0^2 + \left(\tilde{r} - \frac{1}{2}\tilde{\sigma}^2\right)T, \tilde{\sigma}^2 T\right)$$

where $\tilde{\sigma} = 2\sigma$ and $\tilde{r} = 2r + \sigma^2$. Using the expectation in the Black-Scholes formula, we have

$$E[(S_T^2-K)^+] = e^{\tilde{r}T}C(S_0^2,K,t,T;\tilde{\sigma},\tilde{r})$$

so the price of the option is

$$V = e^{-rT}e^{\tilde{r}T}C(S_0^2,K,t,T;\tilde{\sigma},\tilde{r}) = e^{(r+\sigma^2)T}C(S_0^2,K,t,T;\tilde{\sigma},\tilde{r}).$$

10. Intuitively, we know that this probability is related to the probability in exercise 8.12, where you started below L and you are wondering about upward moves to touch L. Here you start above L and wonder about downward moves to touch L. They all depend on the absolute distance between the barrier and the starting point. So we can guess the answer to our question is

$$1 - 2N\left(\frac{L-X_0}{\sigma\sqrt{T}}\right)$$

To see this, we note

$$\min X_t \geq L \Longrightarrow -\max(-X_t) \geq L \Longrightarrow \max(-X_t) \leq -L$$
$$\Longrightarrow \max(X_0 - X_t) \leq X_0 - L \Longrightarrow \sigma \max W_t \leq X_0 - L$$

In the last part we recognize that $X_0 - X_t$ is just $-\sigma W_t$, but it is also a Brownian motion and we just write it as σW_t. Finally we have the reflection principle to tell us that

$$P\left\{\max_{t\in[0,T]} W_t \leq \frac{X_0-L}{\sigma}\right\} = 1 - P(\tau < T) = 1 - 2P\left(W_T \geq \frac{X_0-L}{\sigma}\right) = 1 - 2N\left(\frac{L-X_0}{\sigma\sqrt{T}}\right)$$