Monte Carlo Exercises , Due Nov. 20 at 5 p.m.

1. The following Box-Muller algorithm will produce a pair of independent standard normals (X, Y) based on a pair of independent uniformly distributed random numbers (U_1, U_2).

$$X = R\cos\Theta = \sqrt{-2\log U_1}\cos(2\pi U_2)$$
$$Y = R\sin\Theta = \sqrt{-2\log U_1}\sin(2\pi U_2)$$

   To show that (X, Y) are standard normals, we proceed with the following steps:

   (a) Start with (X, Y) being independent standard normals, derive the joint distribution density function $f(x, y)$ for (X, Y);

   (b) Show that this joint density is axisymmetric, therefore Θ is uniformly distributed in $[0, 2\pi]$, and U_2 is uniformly distributed in $[0, 1]$;

   (c) Use the joint density function $f(x, y)$ to derive the CDF for R

$$P\{R \leq r\} = 1 - e^{-r^2/2}$$

   and use this to show that U_1 is also uniformly distributed in $[0, 1]$.

   Hint: you need to change from Cartesian to polar coordinates when you work on the integrals.

2. Use the MATLAB codes provided to price the following European options. Use 10,000, 50,000, and 100,000 paths to compare the results. Also generate results with antithetic sampling to see if there is indeed a benefit.

   (a) A digital call with strike $K = 100$;

   (b) A knock-in put with strike $K = 100$, barrier $B = 120$ (the put comes to life only if the stock price goes up to \$120 at certain time before the option expiration, therefore making the final exercise less likely);

   (c) An Asian call with payoff $F = \max(\bar{S}_T - K)$, where $\bar{S}_T$ is the average of stock prices from time 0 to T and $K = 100$.

   We assume that $S_0 = \$98$, $r = 1\%$, $\sigma = 25\%$, and $T = 0.5$, and we can use a time discretization $\Delta t = 1/250$.