Lectures 4-6: Interest Rates and PV Analysis
Time Value of Money and Interest Rates

- Prefer to receive payments sooner, rather than later

- Simple compounding: $1 \rightarrow (1 + r)$ after one time period

- This rate is applied to a time period, and quoted according to the length of the time period (daily, monthly, yearly, etc.)

- Annualized rate: $1 + r\Delta t$, where $r$ is annualized and $\Delta t$ is measured in years

- Compounding: $(1 + \frac{r}{n})^{nt} \rightarrow e^{rt}$
Present Value Analysis

• A dollar paid in the future viewed today: should be discounted

• Stream of cash flows (payments)

• Each payment discounted by a factor \((1 + r)^{-i}\) where \(i\) is the number of time periods leading to the payment date, or \(e^{-rt}\) in continuous compounding where \(t\) is the time to payment date (usually measured in years)

• PV of the stream of c.f. is therefore the sum of discounted cash flows

• In the real world the rate \(r\) is (1) different for different maturity \(i\) (or \(t\)), and (2) is time varying in some random fashion.
PV Analysis

• Cash flow stream: \( \mathbf{a} = (a_1, a_2, \ldots, a_n) \)

• PV of cf stream: \( PV(\mathbf{a}) = \frac{a_1}{1 + r} + \frac{a_2}{(1 + r)^2} + \frac{a_3}{(1 + r)^3} + \cdots + \frac{a_n}{(1 + r)^n} \)

• Can be used to compare two different cash flow streams

• Other ways to compare: Proposition 4.2.1

• The crucial role played by the interest rate: one stream is favored over another based on the interest rate used.
Interest rate examples

• Making deposit for the next 20 years, expect retirement payments of $1000 per month for the 30 years afterwards. How much should we save now? Note what a nonsense this question is since the future rates are anything but known to us at this time.

• Mortgage payment calculation: this is for real as the rate quoted in the mortgage contract is fixed, and we are here to find the monthly payment amount.

• A decomposition of the monthly payment: interest + principal, the interest portion covers the interest on the remaining principal incurred for one month.

• Interest portion high at beginning, reduced to zero at the end.

• Principal reduction initially low, becoming more dominant towards the end.
Rate of return

- Invest $a$, receive $b$: \[ a \rightarrow b : r = \frac{b}{a} - 1 \]

- Quoted in percentage

- Generalized to a stream of returns: $r$ solves the equation $P(r)=0$, where $P$ is

\[
P(r) = -a + \sum_{i=1}^{n} b_i (1 + r)^{-i}
\]

- Making sure unique solution exists

- Guaranteed by all positive returns

- Numerical tools needed to solve for the nonlinear equation
Continuously Varying Interest Rates

- Defining short rate $r(s)$: applying to period $(s, s+h)$ where $h$ is very small

  $1 \rightarrow 1 + r(s)h$

- Announced at time $s$, also called spot or instantaneous interest rate.

- Introduce $D(t)$: the amount you will have on account at time $t$ if you deposit 1 at time 0.

- $D(t)$ grows in time if $r$ is positive.

- For small $h$, using differentials to describe is convenient.
Deriving Equations for $D(t)$

- For small $h$
  \[ D(s + h) - D(s) \approx D(s)r(s)h \]

- Let $h$ go to zero
  \[ D'(s) = D(s)r(s) \]

- Solution
  \[ D(t) = \exp \left\{ \int_0^t r(s) \, ds \right\} \]

- Bond price
  \[ P(t) = \frac{1}{D(t)} = \exp \left\{ - \int_0^t r(s) \, ds \right\} \]

- Face the reality: $r(t)$ is random so stochastic models are needed and bond prices will be given in terms of expectations.
Yield Curve

- Average of the spot interest rate up to $t$

$$\bar{r}(t) = \frac{1}{t} \int_0^t r(s) \, ds$$

- This function of $t$ plotted is called the yield curve.
How Is Yield Curve Used

- One dollar paid in t years, PV is \( e^{-\bar{r}(t)t} \)

- Bond price: how much are you willing to pay for a dollar paid (guaranteed) by the issuer (government or corporate) in t years: \( P(t) \)

- Yield of the bond price: \( -\frac{1}{t} \log P(t) \) (assuming no coupons)

- Yield curve constructed from several bond prices, then smoothly connected

- Shape of the yield curve

- Yield curve changes all the time
Random Interest Rates

- Short rate $r(s)$ revealed at time $s$ only
- Stochastic modeling of $r(s)$ needed
- Interest rate models for $r(s)$ start with applications of Brownian motion
- Bond price

$$P(t) = E \left[ \exp \left\{ - \int_0^t r(s) \, ds \right\} \right]$$