Lectures 2 and 3
Examples of call prices in relation to underlying
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Brownian Motion

• $X(t), t \geq 0$: a collection of rv’s indexed by t;

• White noise: the increments
Introducing Brownian Motion

• Focusing on returns over any period of time

\[ R_i = \frac{S_{i+1} - S_i}{S_i} \]

• Reasonable to assume \( R_i \) to have normal distribution

\[ R_i \sim N(\mu \Delta t, \beta^2) \quad \text{or} \quad R_i = \mu \Delta t + \epsilon_i \]

• What can we say about \( \epsilon_i \)?

\[ \epsilon_i \sim \beta N(0, 1) \]
Brownian Motion Definition

- $X(t), t \geq 0$ is said to be a Brownian motion with drift $\mu$ and variance $\sigma^2$ if

- $X(0)$ is a given constant

- For all positive $y$ and $t$, $X(t + y) - X(y)$ (the increment) is independent of the process up to time $y$ and has a normal distribution with mean $\mu t$ and variance $\sigma^2 t$.

**Implications:**

- move step by step, each step a normal rv independent of previous steps

- each step with mean and variance proportional to $t$. 

Important properties

- $X(t)$ is continuous:
  \[
  \lim_{h \to 0} (X(t + h) - X(t)) = 0
  \]

- Nowhere differentiable

- To see it as a limiting process of a random walk, we introduce
  \[
  X_i = \begin{cases} 
  1, & \text{head} \\
  -1, & \text{tail}
  \end{cases}
  \]

- with sum till $n$: $Y_n = X_1 + X_2 + \cdots X_n$

- This is called a symmetric random walk if head and tail have the same probability $1/2$
Convergence

• As $\Delta$ gets smaller and smaller,

• $X(t) - X(0)$ converges to a normal random variable

• mean $= \mu t$, variance $= \sigma^2 t$

• the collection of process values over time becomes a Brownian motion process with drift $\mu$ and variance parameter $\sigma^2$

• Theorem 3.2.1: Given $X(t)=x$, the conditional probability law of the collection of prices $X(y), 0 \leq y \leq t$, is the same for all values of $\mu$. 
Random Walk

- let $\Delta$ to be a small time increment

- head probability $p = \frac{1}{2} \left( 1 + \frac{\mu}{\sigma} \sqrt{\Delta} \right)$ and tail probability $1 - p$

- Consider the process

\[ X(n\Delta) = X(0) + \sigma \sqrt{\Delta} \left( X_1 + X_2 + \cdots X_n \right) \]

- or for $n = t/\Delta$

\[ X(t) - X(0) = \sigma \sqrt{\Delta} \sum_{i=1}^{t/\Delta} X_i \]

- We can compute

\[ E[X(t) - X(0)] \quad Var[X(t) - X(0)] \]
Geometric Brownian Motion

- Problem with Brownian motion modeling stock prices:
  - negative values possible
  - price difference over an interval has the same normal distribution no matter what the price at the beginning of the interval

To fix these two problems, consider $S(t) = e^{X(t)}$ where $X(t)$ is a Brownian motion with drift $\mu$ and variance parameter $\sigma^2$

- $S(t)$ is said to be a geometric Brownian motion process

- increment in log: $\log\left(\frac{S(t+y)}{S(y)}\right) \approx R(y)$
Useful results with geometric Brownian motion

- $\sigma$ is called the volatility parameter

- If $S(0) = s$, we can write $S(t) = se^{X(t)}$

- If $X$ is normal,

  $$E[e^X] = \exp \left( E[X] + Var[X]/2 \right)$$

  $$E[S(t)] = se^{\mu t + \frac{1}{2} \sigma^2 t}$$

- In terms of random walk: $S(y + \Delta) = S(y)e^{X(y+\Delta)-X(y)}$, with move factors

  $$u = e^{\sigma \Delta}, \quad d = e^{-\sigma \Delta}$$
The Maximum Variable

- Let \( X(v), v \geq 0, \) be a Brownian motion

- Consider the maximum process \( M(t) = \max_{0 \leq v \leq t} X(v) \)

- What we know about this rv for a fixed \( t \)?

- Theorem 3.4.1 (conditional distribution):

\[
P(M(t) \geq y | X(t) = x) = e^{-2y(y-x)/t\sigma^2}, \ y \geq 0
\]

- Corollary 3.4.1 (unconditional distribution):

\[
P(M(t) \geq y) = e^{-2y\mu/\sigma^2} \Phi \left( \frac{\mu t + y}{\sigma \sqrt{t}} \right) + \Phi \left( \frac{y - \mu t}{\sigma \sqrt{t}} \right)
\]
The Cameron-Martin Theorem (Girsanov)

• Expectation under different measures

• Reflected in different drifts $\mu \neq 0, \mu = 0$

• Sometimes one expectation ($\mu = 0$) is much easier to obtain than the other

• Theorem 3.5.1: Let $W$ be a rv whose value is determined by $X$ (BM) up to $t$.

$$E_\mu[W] = e^{-\mu^2 t/2\sigma^2} E_0 \left[ W e^{\mu X(t)/\sigma^2} \right]$$