Math 5610/6860, Project No.1

1. You might wonder that why we did not spend much time on polynomial root-finding. After all, polynomials are the first functions that became familiar for many of us. One reason as we mentioned in class is that polynomials of high degree are not very practical in dealing with real world problems, and the other reason is that the state-of-the-art method is based on its related eigenvalue problem. In another word, the polynomial root-finding is recast as an eigenvalue problem and the best eigenvalue algorithm is therefore used. To show the connection between the polynomial root-finding and the eigenvalue problem, we need to revisit some concepts in eigenvalues.

   (a) Derive the companion matrix of the polynomial

   $$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

   (b) Show that the roots of $P(x)$ correspond to the eigenvalues of the companion matrix.

2. One unusual application of Newton's method is to find the reciprocal of a without division by applying Newton's method to solve

   $$f(x) = a - \frac{1}{x}$$

   Here we assume $a \in [0.5, 1]$ and the exact solution $p = 1/a$.

   (a) What is the iteration formula $g(x)$? Why is it a useful algorithm many years ago to compute the reciprocal?

   (b) Use the fix-point theorem to suggest an initial guess p_0.

   (c) We can also use another approach to find the requirement on p_0: the error analysis of Newton's method. First we define the residual

   $$r_n = 1 - a p_n,$$

   and the error of approximation

   $$e_n = \frac{1}{a} - p_n = \frac{r_n}{a}, \quad n = 0, 1, \ldots$$

   Show that $r_{n+1} = r_n^2$ and $e_{n+1}/p = (e_n/p)^2$, then

   $$\left| \frac{e_n}{p} \right| = (r_0)^{2^n}.$$

   (d) Use the last equation above to derive a condition for p_0 so that the iterations will converge to p.

3. Suppose that you have a convergent sequence $\{p_n\}_{n=0}^N$ where $|p_N - p_{N-1}| < TOL$ and you are ready to declare that p_N is the approximating solution for the equation $f(x) = 0$. How do you estimate the order of convergence?

4. (For those registered in 6860) The standard method in practice (as used in MATLAB function `fzero`) is based on the Brent-Dekker method. Use library/online resources to give a brief summary of the method.