

Examples of Discrete RVs Sections 3.4-6

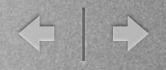
- Binomial Distribution
- Hypergeometric Distribution
- Poisson Distribution

Common Features

- Experiments involving trials, each with "S" or "F" (dichotomous)
- X = total number of "S", takes on 0,1,2,...
- Trials may or may not be independent
- Approximations may be involved



- A class of experiments with following requirements:
 - consists of n (fixed) trials,
 - each with two possible outcomes (S or F),
 - trials are independent,
 - probability of S is constant from trial to trial.



Importance of Verification

- independence of trials
- with of without replacement
- sizes N (population) and n (sample size) matter
- possible approximation (n at most 5% of N)

pmf of binomial rv

- If verified to be binomial
- a sample of n trials
- X=number of "S" among the n trials
- pmf P(X = x) = b(x; n, p)

•
$$b(x;n,p) = {\binom{n}{x}} p^x (1-p)^{n-x} \quad x = 0, 1, \dots, n$$



Example

• Problem 49, page 114

• "S" = cosmetic flaws ("second"), p=0.1

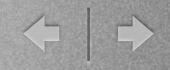
• a. n=6,
$$P(X = 1) = \binom{6}{1} \times 0.1^1 \times 0.9^5 = 0.3543$$

• b. $P(X \ge 2) = 1 - P(X = 0) - P(X = 1)$
 $= 1 - 0.9^6 - \binom{6}{1} \times 0.1^1 \times 0.9^5$
 $= 0.1142$

• C.

1

 $b(0;4,0.1) + b(1;4,0.1) \cdot 0.9 = 0.9185$



mean and variance of binomial rv

- If X is a binomial rv with parameters n and p $X \sim {\rm Bin}(n,p)$
- Expected value and variance

E(X) = npV(X) = np(1-p)

Hypergeometric Distribution

- No replacement
- N individuals: M successes, N-M failures
- Sample n individuals (n<=N, but may be >M)
- X=number of "S" in n individuals
- X is said to have hypergeometric distribution

pmf of hypergeometric dist

$$P(X = x) = h(x; n, M, N) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$$

where x must satisfy

$$\max(0, n - N + M) \le x \le \min(n, M)$$

If n>N-M, we are guaranteed n-N+M "S"s, and on the other hand x cannot exceed M

Explanations:

To arrange, first choose x from M "S"s, and then choose n-x from the rest. The total number of different ways is N choose n.



mean and variance of h dist

$$E(X) = n \cdot \frac{M}{N}, \quad V(X) = \left(\frac{N-n}{N-1}\right) n \frac{M}{N} \left(1 - \frac{M}{N}\right)$$

Compare with the binomial, when N is very large compared to n:

$$\frac{M}{N} \to p, \quad \frac{N-n}{N-1} \to 1$$

Example

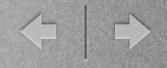
- Problem 69, page 120
- I2 refrigerators (N=I2), 7 defective (M=7), first 6 to be examined (n=6)
- **possible x:** $\max(0, n N + M) = 1$, $\min(n, M) = 6$.

• **a.**
$$P(X = 5) = \frac{\binom{7}{5}\binom{5}{1}}{\binom{12}{6}}$$

- **b.** $P(X \le 4) = P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)$ = 1 - P(X = 5) - P(X = 6)
- **C.** $P\left(X-6\cdot\frac{7}{12}>\sqrt{\frac{6}{11}\cdot 6\cdot\frac{7}{12}\cdot\frac{5}{12}}\right) = P(X \ge 5) = 1 P(X \le 4)$

Poisson Distribution

- originally derived from theory
- model many real world experiments in approximation
- behind Poisson process that models many important natural phenomena



Definition

• X is said to have a Poisson distribution with parameter $\lambda > 0$ if the pmf is

$$p(x;\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, 2, \dots$$



$$\sum_{x=0}^{\infty} p(x;\lambda) = 1$$

 ∞



mean and variance

• Poisson as a limit for binomial as $n \to \infty, p \to 0$:

$$b(x; n, p) \to p(x; \lambda) \text{ if } np \to \lambda$$

- Approximation may become useful
- If X has a Poisson distribution with parameter λ $E(X) = V(X) = \lambda$



Example

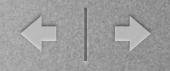
• Problem 81, page 125

- **a.** $P(X \le 10) = e^{-20} \sum_{x=0}^{10} \frac{20^x}{x!}$ **b.** $P(X > 20) = 1 e^{-20} \sum_{x=0}^{19} \frac{20^x}{x!}$
- **d.** $P(|X 20| < 2 \cdot \sqrt{20}) = P(11 < X < 28)$

Poisson Process

- Concerned with the number of events within a time period of length t
- Assumptions:
 - probability that exactly one occurred within short period
 - probability of more than one event is of higher order $\,lpha\Delta t$
 - memory-less: the number of events during time interval is independent of prior events





Poisson Process

 number of events during a time interval of length t is a Poisson rv with parameter

$$\lambda = \alpha t$$