Examples of Discrete RVs
Sections 3.4-6

- Binomial Distribution
- Hypergeometric Distribution
- Poisson Distribution

Common Features

- Experiments involving trials, each with “S” or “F” (dichotomous)
- X = total number of “S”, takes on 0,1,2,...
- Trials may or may not be independent
- Approximations may be involved

Binomial Distribution

- A class of experiments with following requirements:
  - consists of n (fixed) trials,
  - each with two possible outcomes (S or F),
  - trials are independent,
  - probability of S is constant from trial to trial.

Importance of Verification

- independence of trials
- with of without replacement
- sizes N (population) and n (sample size) matter
- possible approximation (n at most 5% of N)

pmf of binomial rv

- If verified to be binomial
- a sample of n trials
- X=number of "S" among the n trials
- pmf $P(X = x) = b(x; n, p)$
- $$b(x; n, p) = \binom{n}{x} p^x (1-p)^{n-x} \quad x = 0, 1, \ldots, n$$

Example

- Problem 49, page 114
- "S" = cosmetic flaws ("second"), p=0.1
- a. n=6, $P(X=1) = \binom{6}{1} \times 0.1^1 \times 0.9^5 = 0.3543$
- b.
$$\begin{aligned} P(X \geq 2) &= 1 - P(X=0) - P(X=1) \\ &= 1 - 0.9^6 - \binom{6}{1} \times 0.1^1 \times 0.9^5 \\ &= 0.1142 \end{aligned}$$
- c.
$$b(0; 4, 0.1) + b(1; 4, 0.1) \cdot 0.9 = 0.9185$$

mean and variance of binomial rv

- If X is a binomial rv with parameters n and p

$$X \sim \mathrm{Bin}(n, p)$$

- Expected value and variance

$$\begin{aligned} E(X) &= np \\ V(X) &= np(1-p) \end{aligned}$$

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Hypergeometric Distribution

- No replacement
- N individuals: M successes, N-M failures
- Sample n individuals (n<=N, but may be >M)
- X=number of “S” in n individuals
- X is said to have hypergeometric distribution

pmf of hypergeometric dist

$$P(X = x) = h(x; n, M, N) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$$

where x must satisfy

$$\max(0, n - N + M) \leq x \leq \min(n, M)$$

If n>N-M, we are guaranteed n-N+M "S"s, and on the other hand x cannot exceed M

Explanations:
To arrange, first choose x from M "S"s, and then choose n-x from the rest. The total number of different ways is N choose n.

mean and variance of h dist

$$E(X) = n \cdot \frac{M}{N}, \quad V(X) = \left(\frac{N-n}{N-1}\right) n \frac{M}{N} \left(1 - \frac{M}{N}\right)$$

Compare with the binomial, when N is very large compared to n:

$$\frac{M}{N} \to p, \quad \frac{N-n}{N-1} \to 1$$

Example

- Problem 69, page 120
- 12 refrigerators (N=12), 7 defective (M=7), first 6 to be examined (n=6)
- possible x: $\max(0, n - N + M) = 1, \quad \min(n, M) = 6.$
- a. $P(X = 5) = \frac{\binom{7}{5}\binom{5}{1}}{\binom{12}{6}}$
- b. $$\begin{aligned} P(X \leq 4) &= P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) \\ &= 1 - P(X = 5) - P(X = 6) \end{aligned}$$
- c. $P\left(X - 6 \cdot \frac{7}{12} > \sqrt{\frac{6}{11} \cdot 6 \cdot \frac{7}{12} \cdot \frac{5}{12}}\right) = P(X \geq 5) = 1 - P(X \leq 4)$

Poisson Distribution

- originally derived from theory
- model many real world experiments in approximation
- behind Poisson process that models many important natural phenomena

Definition

- X is said to have a Poisson distribution with parameter $\lambda > 0$ if the pmf is

$$p(x;\lambda) = \frac{e^{-\lambda}\lambda^{x}}{x!}, \qquad x = 0, 1, 2, \ldots$$

- check $$\sum_{x=0}^{\infty} p(x;\lambda) = 1$$

mean and variance

- Poisson as a limit for binomial as $n \to \infty, p \to 0$:

$$b(x; n, p) \to p(x; \lambda) \quad \text{if} \quad np \to \lambda$$

- Approximation may become useful
- If X has a Poisson distribution with parameter λ

$$E(X) = V(X) = \lambda$$

Example

- Problem 81, page 125
- a. $P(X \leq 10) = e^{-20} \sum_{x=0}^{10} \frac{20^x}{x!}$
- b. $P(X > 20) = 1 - e^{-20} \sum_{x=0}^{19} \frac{20^x}{x!}$
- d. $P\left(|X - 20| < 2 \cdot \sqrt{20}\right) = P(11 < X < 28)$

Poisson Process

- Concerned with the number of events within a time period of length t
- Assumptions:
  - probability that exactly one occurred within short period
  - probability of more than one event is of higher order $\alpha \Delta t$
  - memory-less: the number of events during time interval is independent of prior events

Poisson Process

- number of events during a time interval of length t is a Poisson rv with parameter

$$\lambda = \alpha t$$