MATH 2280-1, Fall 2012
Course Review Sheet

Summary
In this course, we introduced the basic concepts of ordinary differential equations, and discussed several tools in solving mostly linear equations and systems of equations. For nonlinear equations, qualitative behaviors, such as the stability near critical points, are emphasized. The discussion on partial differential equations is kept at minimum, and the focus is on the significance of the three prototype equations (heat equation, wave equation, and Laplace equation) in fundamental sciences.
In the following, we list major topics for each chapter covered in the semester, and clarify on the scope of the final.

Topics:

- Chapter 1: Introduction

  First we described a differential equation as a condition that relates the unknown function and its derivatives, and showed some examples where straightforward integration could lead to a solution. There is a geometric interpretation (slope fields and solution curves) to a differential equation which allows us to visualize solutions, and we should take advantage of various computer softwares to help us to obtain this information. Also it is necessary for us to determine the order of an equation, whether an equation is linear or nonlinear, homogeneous or nonhomogeneous. In the case that straightforward integration is possible to lead to a solution, we realize that the success is dependent on whether we can separate the variables in the equation. This suggests the first thing to try when we encounter a new equation.

  As a general rule of thumb with equations, keep in mind that before you claim a solution, you should always confirm by verifying if your answer satisfies the original equation and various conditions (such as initial and/or boundary conditions).

  On the theoretical side, it should be pointed out that existence and uniqueness of the solutions are not automatically guaranteed. Theorem 1 on page 24 gives the conditions to guarantee the existence and uniqueness of a *local* solution for the general initial value problem. We note that this theorem can be extended to system of equations.

- Chapter 2: Mathematical Models

  We only use the population dynamics equations as examples to demonstrate some of the origins of differential equations. The common theme that appears constantly in these models is a rate of change on the left-hand-side, and the resulting effects on the righ-hand-side. These population equations are nonlinear equations, and solutions usually include some exponential terms, so the solution behavior as $t \to \infty$ (asymptotes) is of crucial interest. We should watch out for the sign in front of t (growth vs. decay) in those exponential terms, as they will dominate the long time behavior of the solutions.

For most practical problems, numerical solutions are the most efficient way, even though only approximate solutions are generated. The Euler method is the most fundamental method for which we can even use a simple calculator to plot a few points on the approximate solution curve. Improved methods are all based on the Euler method, and it would be sufficient for us to know how to implement Euler method for one or two steps for this final.

- Chapter 3: Linear Equations of Higher Order

  First we must be able to determine if an equation is linear or not, and in the linear case we expect to obtain a generic solution. Before we solve the equations, we should explore the properties of linear equations. For homogeneous linear equations, the greatest advantage is that a linear combination of solutions is still a solution. This is called the principle of superposition. To organize all possible solutions and find the one that satisfies the extra condition that comes with the problem (such as an initial condition), we need to introduce the concept of general solutions. An intuitive way is to select a collection that is supposed to "cover" all possible solutions, and we need the concept of linearly independent solutions. This is really the parallel of linear independence of vectors in linear algebra. There is a formal definition, similar to the definition in linear algebra, and there is also another one based on the Wronskian that is more straightforward to use. For an n-th order linear equation, we expect to find n linearly independent solutions and a general solution can be formed by taking a linear combination of these n linearly independent solutions.

  Constant coefficient equations are a special kind of linear equations. In this case the problem is reduced to an eigenvalue problem, and the solution behavior is determined by the roots of the characteristic equation. There are details relating to the nature of the roots, whether they are repeated, complex or real, and they contribute to the characterization of the solution behavior. The examples of mass-spring systems, mechanical vibration, and electrical circuits will help us to appreciate the significance of the second-order constant coefficient equations.

  Solutions to nonhomogeneous equations are related to the solutions to the corresponding homogeneous equations. In particular, a solution to the nonhomogeneous equation can be written as

  $$y = y_c + y_p$$

  where y_c is a general solution to the corresponding homogeneous equation, and y_p is any particular solution. We discussed two approaches to find a particular solution: the method of undetermined coefficient, and the variation of parameters. One feature that is important in practice is the phenomenon of resonance, which can be inspected by comparing the frequency of the forcing term and the frequency of the system.

- Chapter 4: System of Differential Equations

  It turns out that an equation of any order can be recast as a first-order system, so it is only necessary for us to consider these first-order systems. Various matrix techniques are naturally used to develop methods to solve system of equations. The most obvious one is the method of elimination, but we will learn later that there are more efficient methods.

- Chapter 5: Linear Systems of Differential Equations

  The general form is $\mathbf{x}' = A\mathbf{x}$ and the matrix A is the focus. More specifically, the eigenvalues and eigenvectors of A will play the major role. If there is a complete set of eigenvalues/eigenvectors, a general solution to the homogeneous equation will be

  $$\mathbf{x} = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} + \ldots + c_n\mathbf{v}_n e^{\lambda_n t}$$

  In cases where we do not have n linearly independent eigenvectors, complications arise and we will need to address them more carefully by introducing so-called generalized eigenvectors. It is not our priority to cover the general situation in the final.

  The mass-spring system can be used again to illustrate the roles played by the eigenvalues, particularly when we realize that the eigenvalues are complex and the periodic behavior associated with the complex eigenvalues.

  We also introduced matrix exponentials, the connection with fundamental matrix solutions, and the representation of solutions in these matrix exponentials. A fundamental matrix contains all the independent solution vectors and it is therefore invertible, and any solution (to the homogeneous equation) can be expressed in terms of a fundamental matrix.

- Chapter 6: Nonlinear Systems and Phenomena

  It is not realistic to expect closed-form solutions for nonlinear equations in most cases. Instead we focus on the behavior of the solution near the critical points, and try to connect solution curves between all these critical points. Near a critical point, the solutions of the nonlinear equation behave quite close to that of the corresponding linear system, which is derived by calculating the Jacobian of the right-hand-side of the nonlinear equations, evaluated at the critical point in consideration. Therefore the eigenvalues of the Jacobian matrix will determine the behavior of the so-called almost linear systems, except in the borderline cases (zero real part eigenvalues). For the ecological models, we should be able to identify the nature of the system the equations describe, whether it is predator, or competitor model.

- Chapter 7: Laplace Transform

  Laplace transform can be efficient in solving some initial value differential equation problems, but we should be aware that success is not guaranteed and the complication with the improper integrals can cause problems.

  Transform methods can be quite powerful in dealing with constant coefficient differential equations. The power of a transform is in its ability to transform a complicated equation into a relatively less complicated equation. However the task of transforming back the solution sometimes overweights the convenience. For the final, Laplace transform formulas beyond those in Figure 7.1.2 will be provided during the exam.

- Chapter 9: Fourier Series Methods

  Fourier series are introduced to "standardize" all piecewise smooth periodic functions, in the sense that we can just use an infinite collection of coefficients to specify a periodic function. The coefficient formulas are natural to memorize: cosine coefficients are given by an integral involving the function to be worked on ($f(x)$) and the cosine function, and similar formulas for the sine coefficients. The reason that they can be so useful in partial differential equations is that there are those modes arising from a problem, and each one of them corresponds to one such trigonometric term. If we understand the behavior of one mode and we can decompose the initial condition into a sum of these modes, the total solution can be expressed as a Fourier series, with a little modification that the coefficients now depend on another variable. Different equations will have different variable coefficients in these Fourier series. To solve a PDE using Fourier series, we must make sure that the domain is rectangular (corresponding to these natural modes, or eigenfunctions), we can expand f into a Fourier series, and we know what coefficients that need to be in front of the sine and cosine terms.

  The Laplace equation was not discussed in class, so it will not be included in this exam.