Midterm 2 Practice Problems Answers, Math 2280, Fall 2012

1. Introduce $x_1 = y, x_2 = y', x_3 = y''$, we have the following system

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3t & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \sin t \end{bmatrix}$$

2. The equation for x is

$$[(D-1)(D+5)+18]\, x = 0,$$

or

$$x'' + 4x' + 13x = 0.$$

The solutions to the characteristic equation are $\lambda = -2 \pm 3i$. Therefore the general solution can be written as

$$x = e^{-2t}\,(a_1 \cos 3t + a_2 \sin 3t)$$
$$y = e^{-2t}\,(b_1 \cos 3t + b_2 \sin 3t)$$

By plugging into one of the equations, say $x' = x + 9y$, we have

$$e^{-2t}\,[(-2a_1 + 3a_2)\cos 3t + (-2a_2 - 3a_1)\sin 3t] = e^{-2t}\,[(a_1 + 9b_1)\cos 3t + (a_2 + 9b_2)\sin 3t]$$

Matching coefficients of $\cos 3t$ and $\sin 3t$, we get

$$a_1 = -\frac{3}{2}(b_1 + b_2)$$
$$a_2 = \frac{3}{2}(b_1 - b_2)$$

3. The eigenvalues are 0 and -3 and the corresponding eigenvectors are $[1, 2]^T$ and $[1, -1]^T$, so a general solution is

$$\mathbf{x} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t},$$

and after plugging the initial conditions we have $c_1 = 5$ and $c_2 = 0$.

4. Since

$$B = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \quad B^2 = 0,$$

Taylor's expansion gives

$$e^{Bt} = I + Bt,$$

and

$$e^{At} = e^{2t}e^{Bt} = \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix} e^{2t}.$$

The solution to the nonhomogeneous equation is

$$\mathbf{x} = e^{At}\int_0^t e^{-As} f\, ds = \begin{bmatrix} 1 \\ 0 \end{bmatrix} (e^{2t} - e^t)$$

5. (a) The linearized system is

$$u' = 6u - 5v$$
$$v' = 2u - v$$

(b) The Jacobian is

$$J = \begin{bmatrix} 6 & -5 \\ 2 & -1 \end{bmatrix}$$

and the eigenvalues are 1 and 4, both positive, so the linear system has a unstable improper node at this critical point. This is not one of those borderline cases so the nonlinear system will also have an unstable improper node.