Solution Notes for Homework Assignment 10

5.6.3. Try

$$\mathbf{x}_p = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} t^2 + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} t + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

We have

$$2 \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} t + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 3a_1 + 4a_2 \\ 3a_1 + 2a_2 + 1 \end{bmatrix} t^2 + \begin{bmatrix} 3b_1 + 4b_2 \\ 3b_1 + 2b_2 \end{bmatrix} t + \begin{bmatrix} 3c_1 + 4c_2 \\ 3c_1 + 2c_2 \end{bmatrix}$$

So we have following 3 systems to solve

$$\begin{aligned} 3a_1 + 4a_2 &= 0 \\ 3a_1 + 2a_2 &= -1 \end{aligned}$$

$$\begin{aligned} 3b_1 + 4b_2 &= 2a_1 \\ 3b_1 + 2b_2 &= 2a_2 \end{aligned}$$

$$\begin{aligned} 3c_1 + 4c_2 &= b_1 \\ 3c_1 + 2c_2 &= b_2 \end{aligned}$$

The first system should be solved first, and then the second, then finally the third. We have solutions

$$a_1 = -\frac{2}{3},\ a_2 = \frac{1}{2},\ b_1 = \frac{10}{9},\ b_2 = -\frac{7}{6},\ c_1 = -\frac{31}{27},\ c_2 = \frac{41}{36}.$$

5.6.6. Try

$$\mathbf{x}_p = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} te^t + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} e^t.$$

5.6.9. Notice that the eigenvalues of the homogeneous system are $\pm 2i$, so the general solution of the homogeneous system has the following form

$$\mathbf{x}_c = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$$

where

$$\begin{aligned} \mathbf{x}_1(t) &= \mathbf{a}\cos 2t - \mathbf{b}\sin 2t \\ \mathbf{x}_2(t) &= \mathbf{b}\cos 2t + \mathbf{a}\sin 2t \end{aligned}$$

and $\mathbf{a} \pm \mathbf{b}$ are the complex eigenvectors.

If we try

$$\mathbf{x}_p = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \cos 2t + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \sin 2t + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} t \cos 2t + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} t \sin 2t$$
$$= \mathbf{a} \cos 2t + \mathbf{b} \sin 2t + \mathbf{c}\, t \cos 2t + \mathbf{d}\, t \sin 2t$$

we should expect that solutions for a_1, a_2, b_1 and b_2 are not unique.

To see how the solution in the text is obtained, we use the vector notation in the comparison of coefficients

$$\mathbf{d} - 2\mathbf{a} = A\mathbf{b}, \ \mathbf{c} + 2\mathbf{b} = A\mathbf{a} + \mathbf{f}, \ -2\mathbf{c} = A\mathbf{d}, \ 2\mathbf{d} = A\mathbf{c}.$$

If we choose $\mathbf{a} = 0$, then we get

$$\mathbf{b} = \frac{1}{4}\mathbf{f}, \ \mathbf{c} = \frac{1}{2}\mathbf{f}, \ \mathbf{d} = \frac{1}{4}A\mathbf{f},$$

and here $\mathbf{f} = [1, 0]^T$.

5.6.17.
$$\mathbf{x} = e^{At} \int_0^t e^{-As} \mathbf{f}(s)\, ds = e^{At} \begin{bmatrix} \int_0^t 10(-e^s + 7e^{-5s}) + 15(7e^s - 7e^{-5s})\, ds \\ \int_0^t 10(-e^s + e^{-5s}) + 15(7e^s - e^{-5s})\, ds \end{bmatrix}$$

5.6.23. Similar to 5.6.17.

5.6.27. Similar to 5.6.17.

6.1.2. Figure 6.1.16

6.1.5. Figure 6.1.12.

6.1.10. Consider the system
$$x_1' = x_2, \ x_2' = -x_1 - 4x_1^3 - 2x_2$$

6.1.14. We can solve each equation by itself.

6.1.20. The characteristic equation is $\lambda^2 + 4\lambda + 5 = 0$.

6.1.24. We have $dy/dx = x/y$.

6.2.2. Two distinct real eigenvalues 3 and 2.

6.2.8. Complex eigenvalues $-2 \pm 3i$.

6.2.12. Two distinct eigenvalues 3 and 2.

6.2.15. Complex eigenvalues $-1 \pm i$.

6.2.22. The Jacobian is
$$J = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}$$