# Solution Notes for Homework Assignment 4

3.1.3 As we form a linear combination,

$$y = c_1 \cos 2x + c_2 \sin 2x, \quad y' = -2c_1 \sin 2x + 2c_2 \cos 2x$$

and after we plug in $x = 0$ in both, and set them to the initial conditions, we have

$$c_1 = 3, \quad 2c_2 = 8$$

so the solution to the IVP is $y = 3\cos 2x + 4\sin 2x$.

3.1.6

$$y = c_1 e^{2x} + c_2 e^{-3x}, \quad y' = 2c_1 e^{2x} - 3c_2 e^{-3x},$$

The system for $c_1$ and $c_2$ is

$$c_1 + c_2 = 7, \quad 2c_1 - 3c_2 = -1$$

and the solutions are $c_1 = 4$ and $c_2 = 3$.

3.1.10

$$y = c_1 e^{5x} + c_2 x e^{5x}, \quad y' = 5c_1 e^{5x} + c_2 e^{5x} + 5c_2 x e^{5x} = (5c_1 + c_2 + 5c_2 x)e^{5x}$$

The system for $c_1$ and $c_2$ is

$$c_1 = 3, \quad 5c_1 + c_2 = 13$$

so $c_1 = 3$ and $c_2 = -2$.

3.1.14

$$y = c_1 x^2 + c_2 x^{-3}, \quad y' = 2c_1 x - 3c_2 x^{-4}$$

so

$$y(2) = 4c_1 + \frac{c_2}{8} = 10, \quad y'(2) = 4c_1 - \frac{3c_2}{16} = 15$$

so the solutions are $c_1 = 3$ and $c_2 = -16$.

3.1.23

$$f(x)/g(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

and the ratio is not a constant on the real line. Therefore they are linearly independent on the real line.

3.1.24

$$g(x) = 1 - \cos 2x = 1 - (\cos^2 x - \sin^2 x) = 2\sin^2 x = 2f(x)$$

so they are linearly dependent.

3.1.35 The characteristic equation is $r^2 + 5r = r(r+5) = 0$ and the roots are $r_1 = 0$ and $r_2 = -5$. A general solution is

$$y = c_1 + c_2 e^{-5x}$$

3.1.40 The characteristic equation is $9r^2 - 12r + 4 = (3r-2)^2 = 0$. We have a double root $r = 2/3$ so a general solution is

$$y = e^{2x/3}(c_1 + c_2 x)$$

3.1.45 This involves a double root $r = -10$ so the characteristic equation should read $(r+10)^2 = r^2 + 20r + 100 = 0$. The original ODE should be

$$y'' + 20y' + 100y = 0$$

3.2.2

$$5c_1 + c_2(2 - 3x^2) + c_3(10 + 15x^2) = (5c_1 + 2c_2 + 10c_3) + (-3c_2 + 15c_3)x^2 = 0$$

We will try to find some $c_1, c_2, c_3$, not all zero, such that

$$5c_1 + 2c_2 + 10c_3 = 0, \quad -3c_2 + 15c_3 = 0$$

We know that we have two equations and three unknowns, and the linear algebra theory tells us that we can find non-zero solutions. Therefore, these functions are linearly dependent on the real line.

3.2.5 We notice $\cos 2x = 2\cos x - 1$, therefore if we require

$$17c_1 + c_2 \cos^2 x + c_3 \cos 2x = 17c_1 - c_3 + (c_2 + 2c_3)\cos^2 x = 0,$$

we only need to find $c_1, c_2, c_3$ such that $17c_1 - c_3 = 0$ and $c_2 + 2c_3 = 0$. Again, we have two equations, three unknowns, and we can find non-zero solutions. These functions are also linearly dependent.

3.2.8

$$W = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = e^x e^{2x} e^{3x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = 2e^{6x} \neq 0 \quad \text{for all } x$$

3.2.11

$$W = \begin{vmatrix} x & xe^x & x^2e^x \\ 1 & (1+x)e^x & (2x+x^2)e^x \\ 0 & (2+x)e^x & (x^2+4x+1)e^x \end{vmatrix} \neq 0$$

3.2.14 Assume

$$y = c_1e^x + c_2e^{2x} + c_3e^{3x}, \quad y' = c_1e^x + 2c_2e^{2x} + 3c_3e^{3x}, \quad y'' = c_1e^x + 4c_2e^{2x} + 9c_3e^{3x}$$

so we need to solve the linear system

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ c_1 + 2c_2 + 3c_3 &= 0 \\ c_1 + 4c_2 + 9c_3 &= 3 \end{aligned}$$

The solutions are $c_1 = 3/2, c_2 = -3, c_3 = 3/2$.

3.2.22

$$y = c_1e^{2x} + c_2e^{-2x} - 3, \quad y' = 2c_1e^{2x} - 2c_2e^{-2x},$$

therefore

$$c_1 + c_2 = 3, \quad 2c_1 - 2c_2 = 10$$

We solve to obtain $c_1 = 4, c_2 = -1$.

3.2.27 We require

$$c_1 + c_2x + c_3x^2 = 0$$

for all $x$. Differentiate it once, we have

$$c_2 + 2c_3x = 0,$$

and twice,

$$2c_3 = 0.$$

The last condition sets $c_3 = 0$, which in turn sets $c_2 = 0$ based on the second last equation. The first equation then has $c_1 = 0$.

3.2.31 (a) Since $y''(a) = -py'(a) - qy(a)$ from the equation, we cannot impose a condition on it separately.

(b) If $y(0) = 1, y'(0) = 0$, then according to the equation, $y''(0) = 2y'(0) + 5y(0) = 0 + 5 = 5$. Therefore $C = 5$.