Solution Notes for Homework Assignment 3

2.2.4

$$f(x) = 3x - x^2 = x(3 - x) = 0, \text{ so } x = 0, 3.$$

From the left, f starts negative, then becomes positive once it crosses $x = 0$, and then becomes negative again after $x = 3$. We can see that $x = 0$ is unstable and $x = 3$ is stable.

2.2.7 $f(x) = (x - 2)^2$ so $x = 2$ is the only critical point, the sign is positive away from this point so it is unstable as curves move away from it when $x_0 > 2$.

2.2.11 $f(x) = (x - 1)^3$ so $x = 3$ is the only critical point, the sign is positive $x_0 > 1$ and negative otherwise, so it is unstable as curves are leaving in both cases.

2.2.19

$$f(x) = x(10 - x) - 10h = -x^2 + 10x - 10h = 0, \quad x = \frac{1}{2}\left(10 \pm \sqrt{100 - 40h}\right)$$

If $h < 2.5$ we have two distinct positive roots and they are both legitimate. If $h = 2.5$ there will be only one real root. On the other hand if $h \leq 2.5$ we will only have complex roots which do not make sense in this application.

2.2.21 (a) $f(x) = x(k - x^2)$. If $k \leq 0, k - x^2 \leq 0$ so $x = 0$ is the only critical point. $f > 0$ from the left and $f < 0$ from the right so it is stable.

1. If $k > 0$, we can write $f(x) = -x(x - \sqrt{k})(x + \sqrt{k})$, When $x < -\sqrt{k},\ f > 0$, and it turns to negative after $x = -\sqrt{k}$, so $c = -\sqrt{k}$ is a stable critical point, but then it changes from negative to positive at $x = 0$, and then from positive to negative again at $x = \sqrt{k}$ so $x = \sqrt{k}$ is also stable.

2.3.2 (a)

$$\frac{dv}{v} = -kdt, \quad \log|v| = -kt + C, \quad v = Ae^{-kt} = v_0 e^{-kt}$$

$$x(t) = x_0 + \int_0^t v(s)\, ds = x_0 + \frac{v_0}{k}\left(1 - e^{-kt}\right)$$

(b) From the solution v we see that v does not change sign, and from $x(t)$ we see that as $t \to \infty,\ x(t) \to x_0 + v_0/k$.

2.3.6

$$\frac{dv}{v^{\frac{3}{2}}} = -kdt, \quad -2v^{-1/2} = -kt + C, \quad v = \frac{1}{(\frac{kt}{2} + \frac{C}{2})^2}.$$

Using $v(0) = v_0$, we have

$$v(t) = \frac{1}{\left(\frac{1}{2}kt + \frac{1}{\sqrt{v_0}}\right)^2}$$

The argument is similar to problem 2 that as $t \to \infty$, $x \to x_0 + \frac{2}{k}\sqrt{v_0}$.

2.3.9 We write the equation as

$$v' + 0.1v = 5,$$

and obtain the solution

$$v(t) = 50 + Ce^{-0.1t}$$

As $t \to \infty$, $v \to 50$.

2.3.12 The parameters are

$$W = 640(lb), \quad m = W/g = 640/32 = 20(slug), \quad B = 62.5 \times 8 = 500(lb), \quad F_R = -v$$

and the equation is

$$\frac{dv}{dt} = -7 - \frac{v}{20}$$

and the solutions are

$$v(t) = 140(e^{-t/20} - 1), \quad y(t) = 140\left(20(1 - e^{-t/20}) - t\right)$$

notice that $v < 0$ points downward. We solve for t^* such that $v(t^*) = -75$, which we obtain $t^* = 15.35$ (s) and

$$y(t^*) \approx -648 \quad (ft)$$

2.3.20 The equation is

$$\frac{dv}{dt} = -g - \frac{v^2}{800},$$

and the solution satisfying the initial condition is

$$v(t) = 160 \tan\left(\frac{\pi}{4} - \frac{t}{800}\right), \quad y(t) = 800\left[\log\left(\cos\left(\frac{\pi}{4} - \frac{t}{800}\right)\right) - \log(\cos\frac{\pi}{4})\right]$$

$v = 0$ when $t = 800 \times \pi/4$ so the maximum height is $y(200\pi) \approx 277$ ft.

2.3.25 (a) From page 107,

$$v^2 = v_0^2 + 2GM\left(\frac{1}{r} - \frac{1}{R}\right),$$

and the projectile is at its maximum height when $v = 0$. Solving for that we have

$$r_{max} = \frac{2GMR}{2GM - Rv_0^2}$$

(b) Plug in $r_{max} = 100$ kilometers to solve for v_0;

(c) Use $v_0 = 0.9\sqrt{\frac{2GM}{R}}$.