Solution Notes for Homework Assignment 2

1.4.3

$$\frac{dy}{y} = \sin x\,dx, \quad \log|y| = -\cos x + C_1, \quad y = Ce^{-\cos x}$$

1.4.6

$$\frac{dy}{\sqrt{y}} = 3\sqrt{x}\,dx, \quad 2\sqrt{y} = 2x^{\frac{3}{2}} + C_1, \quad y = \left(x^{3/2} + C\right)^2$$

1.4.22

$$\frac{dy}{y} = (4x^3 - 1)\,dx, \quad \log|y| = x^4 - x + C_1, \quad y = Ce^{x^4 - x}$$

To determine the constant, we set

$$y(1) = Ce^{1-1} = C = -3,$$

so $y = -3e^{x^4-x}$ is the solution to the initial value problem.

1.4.26

$$\frac{dy}{y^2} = (2x + 3x^2)\,dx, \quad -\frac{1}{y} = x^2 + x^3 + C, \quad y = -\frac{1}{x^2 + x^3 + C}$$

To determine the constant, we have

$$y(1) = -\frac{1}{2 + C} = -1,$$

so $C = -1$ and the solution is $y = -\frac{1}{x^2+x^3-1}$.

1.4.37

$$A(18) = 5000e^{0.08\times 18}$$

1.4.40

$$\tau = \frac{\log 2}{k} = 5.27, \quad k = \frac{\log 2}{\tau} \approx 0.1315$$

To cut the level by a factor of 100, we need t such that

$$A(t) = A_0 e^{-kt} = \frac{1}{100} A_0$$

so $t = \log 100/k \approx 35$ years

1.4.43

$$\frac{dT}{dt} = -kT, \quad T(t) = 25e^{-kt}$$

As $T(20) = 25e^{-20k} = 15,\ k = 0.02554$, we solve for t^* such that

$$T(t^*) = 25e^{-kt^*} = 5$$

so $t^* \approx 63$ minutes.

1.5.5

$$y' + \frac{2}{x}y = 3,$$

so $P = 2/x, Q = 3$.

$$\rho = e^{2\log x} = x^2, \quad \int \rho Q\, dx = x^3 + C$$

and the solution is

$$y = x^{-2}\left(x^3 + C\right), \quad y(1) = 1 + C = 5, \quad C = 4, \quad y = x + 4x^{-2}$$

1.5.14

$$y' - \frac{3}{x}y = x^2, \quad \rho = e^{-3\log x} = x^{-3}, \int \rho Q\, dx = \int x^{-3}x^2\, dx = \log x + C,$$

$$y = x^3\left(\log x + C\right), \quad y(1) = C = 10,$$

the solution is

$$y = x^3\left(\log x + 10\right)$$

1.5.17

$$P(x) = \frac{1}{1+x},\ Q(x) = \frac{\cos x}{1+x}$$

$$\rho = e^{\log(1+x)} = 1 + x, \quad \int \rho Q\, dx = \sin x + C, \quad y = \frac{1}{1+x}\left(\sin x + C\right),$$

so

$$y(0) = C = 1, \quad y = \frac{1}{1+x}(\sin x + 1)$$

1.5.20

$$P(x) = -(1+x), \quad Q(x) = 1 + x, \quad \rho = e^{-x-\frac{1}{2}x^2} \quad \int \rho Q\, dx = -e^{-x-\frac{1}{2}x^2} + C$$

$$y = -1 + Ce^{x+\frac{1}{2}x^2}, \quad y(0) = -1 + C = 0, \quad C = 1.$$

1.5.23

$$P(x) = 2 - \frac{3}{x}, \quad Q(x) = 4x^3$$

$$\rho = e^{2x - 3\log x} = x^{-3}e^{2x}, \quad \int \rho Q\, dx = \int 4e^{2x}\, dx = 2e^{2x}$$

$$y = x^3 e^{-2x}\left(2e^{2x} + C\right) = 2x^3 + Cx^3 e^{-2x}$$

1.5.29

$$P(x) = -2x,\ \ Q(x) = 1,\ \ \rho = e^{-x^2},\ \ y = e^{x^2}\left(\int_0^x e^{-t^2}\, dt + y_0\right) = e^{x^2}\left(\frac{\sqrt{\pi}}{2}\mathrm{erf}(x) + y_0\right)$$

1.5.33

$$V_0 = 1000,\ \ c_i = 0,\ \ r_i = r_o = 5, \quad x_0 = 0.1, \quad \frac{dx}{dt} = -\frac{1}{200}x$$

$$x(t) = 0.1e^{-\frac{t}{200}}$$

To answer the question, we find t^* such that $x(t^*) = 0.01$, the result is

$$t^* = 200\log 10 \approx 460.51(sec)$$

1.5.38

$$P(x) = \frac{5}{200}, \quad Q(x) = \frac{5x}{100}$$

2.1.2

$$\frac{1}{x(10-x)} = \frac{1}{10}\left(\frac{1}{x} + \frac{1}{10-x}\right)$$

2.1.6

$$\frac{1}{x(x-5)} = \frac{1}{5}\left(\frac{1}{x-5} - \frac{1}{x}\right)$$

2.1.11 (a) The equation is

$$\frac{dP}{dt} = \frac{k}{\sqrt{P}}P = k\sqrt{P}$$

and we integrate

$$\frac{dP}{\sqrt{P}} = kdt$$

to obtain the solution.

(b) Using $P_0 = 100$, and $P(6) = 169$, we solve for k in

$$169 = (3k + 10)^2$$

to get $k = 1$, and then

$$P(12) = (6 + 10)^2 = 256$$

2.1.15 From $aP - bP^2 = bP\left(\frac{a}{b} - P\right)$ we know $M = a/b$, but

$$\frac{a}{b} = \frac{B_0/P_0}{D_0/P_0^2} = \frac{B_0 P_0}{D_0}$$

2.1.27

$$\frac{dP}{dt} = (kP - \delta)P = kP\left(P - \frac{0.01}{k}\right), \quad M = 0.01/k$$

The solution is

$$P(t) = \frac{2/k}{200 + \left(\frac{0.01}{k} - 200\right)e^{0.01t}}$$

$P'(0) = 2$ implies

$$200(200k - 0.01) = 2$$

so $k = 10^{-4}$, and

$$P(t) = \frac{20000}{200 - 100e^{0.01t}}$$

(a) Solve for t^* such that $P(t^*) = 1000$, we get $t^* = 100 \log 1.8 \approx 58.78$ months;

(b) Doomday occurs when the denominator vanishes, which is $t = 100 \log 2 \approx 69.31$ months.