Practice Problem Solution Keys

1. (a) First-order, nonlinear, homogeneous.

$$y = \frac{1}{4x+2}$$

(b)

$$\frac{d^2x}{dz^2} + x = 2$$

$$x = A\cos z + B\sin z + 2 = A\cos(\log t) + B\sin(\log t) + 2$$

2. (a)

$$r^2 + 2r + 4 = (r+1)^2 + 3 = 0,\ r = -1 \pm \sqrt{3}\,i$$

This system is underdamped.

(b)

$$x_c = e^{-t}\left(C\cos\sqrt{3}t + D\sin\sqrt{3}t\right),$$

$$x_p = -\frac{5}{61}\cos 3t + \frac{6}{61}\sin 3t.$$

(c) No resonance.

3. (a) For tank 1, the inflow salt rate is $800 \times 1/200 = 4$, and outflow salt rate is $800 \times x/400 = 2x$.

(b)

$$A = \begin{bmatrix} -2 & 0 \\ 2 & -1 \end{bmatrix},\quad \mathbf{f} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\mathbf{x} = -2\begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-2t} - 8\begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t} + \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

(c)

$$\Phi(t) = \begin{bmatrix} e^{-2t} & 0 \\ -2e^{-2t} & e^{-t} \end{bmatrix},\quad e^{At} = \Phi(t)\Phi^{-1}(0) = \begin{bmatrix} e^{-2t} & 0 \\ 2e^{-t} - 2e^{-2t} & e^{-t} \end{bmatrix}$$

4. (a) A predator-prey system

(b) Critical points $(0,0),\ (4,2)$. For $(4,2)$, introduce $u = x-4, v = y-2$, and we obtain the linearized system

$$u' = -8v,\ v' = 2u$$

(c)

$$u = 2D\cos 4t - 2C\sin 4t,\ \ v = C\cos 4t + D\sin 4t.$$

This is periodic in t, for the linearized system.

(d) Borderline case.

5.

$$X = \frac{s}{s^2+1} + \frac{e^{-s}}{s(s^2+1)}$$

$$x = \cos t + u(t-1)\,(1-\cos(t-1))$$

6. $f(x)$ is odd.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = -\frac{2}{\pi}\int_0^{\pi} \sin nx\,dx = \frac{2}{n\pi}\,(\cos n\pi - 1) = \begin{cases} 0 & n \text{ even} \\ -\frac{4}{n\pi} & n \text{ odd} \end{cases}.$$

7.

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-4n^2 t}\sin nx$$