Summary of Chapter 7

In this chapter we introduce the tool of Laplace transform to deal with specifically the initial value problems for linear differential equations. As we see from our discussions in previous chapters, there is a general method to solve systems of linear differential equations based on the eigenvalue approach. However, when it comes to nonhomogeneous problems, we would require the right-hand-side to be an elementary function, or we need to solve another differential equation (variation of parameters). With Laplace transforms, there is a procedure that is easy to follow.

It should be pointed out that due to the specific form of the transform, an explicit knowledge of the initial values of the unknown function is assumed. More importantly, the success of the approach depends on the inverse transform of the unknown function, which may be difficult to obtain. The Laplace transform is just one more tool in the toolbox that provides with an alternative.

1 Laplace Transform

The Laplace transform is a way to take a function in one variable, say t, to another function in s, via an integration operation. It can be viewed as a map from one function space to another. The value of the transform would have been mostly lost if it had not been for the convenience related to differentiation.

First of all, the formal definition is the following: for a function $f(t)$ defined for $t \geq 0$, under technical conditions, we have the Laplace transform

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

It takes one function $f(t)$ and gives another function $F(s)$ via this integration procedure. One obvious requirement is that the improper integral has to converge for the transform to exist. A negligence can cause fatal problems and it is often cited as a difficulty in applying the transform to more general problems.

For most elementary functions, the Laplace transforms are worked out and they are listed in a table on page 446. The linearity of the transform allows us to apply this table in daily applications if we need the transform of a linear combination of these functions. These functions can be categorized as

- polynomials: t^n $(n \geq 0)$ $\longrightarrow n!/s^{n+1}$ $(s > 0)$;
- extension of polynomials to t^a where $a > -1$ is not necessarily an integer, in which case a gamma function is present in the transform;
- exponential functions: $e^{at} \longrightarrow 1/(s-a)$, and hyperbolic sine and cosine;
- trigonometric functions: $\cos kt \longrightarrow s/(s^2+k^2)$, $\sin kt \longrightarrow k/(s^2+k^2)$.

Another function that deserves special mention is the unit step function, sometimes called the Heaviside function, which is particularly important in engineering applications as it stands for the effect of switch-on at certain time.

On the technical side, first we have to assure that the improper integral converges, that depends on both the values of t and s. In general, it is the behavior of f as $t \to +\infty$ that determines the convergence, and we see that with the help of a factor e^{-st} (for $s > 0$), the integrand can be made to decay fast enough, as long as the growth in f does not win the competition with the decay of e^{-st}. This is what is behind the concept of exponential order. On the other hand, since f is involved in an integral, it is not necessary for f to be continuous everywhere, instead we only need f to be *piecewise* smooth, meaning that finitely many jumps with finite jump size are allowed. Again this makes the transform really useful in engineering when signals like impulses are often considered.

2 Laplace Transform in Solving Initial Value Problems

The single most important aspect that makes Laplace transform so useful in differential equations is the fact

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

This can be appreciated in the following interpretation: instead of the action of differentiation in t variable, the corresponding action in s variable is mostly multiplication by a simple factor s. With this property, we are able to transform a differential equation for $x(t)$ into an algebraic equation, or another, hopefully simpler, differential equation for $X(s) = \mathcal{L}\{x(t)\}$. The success of this approach has to wait until the last step, where the inverse transform is called on to turn $X(s)$ back into $x(t)$. There is no set procedures that is always successful to find the inverse transform. It is very much experience based, with the clear goal to represent $X(s)$ in a form that we can identify the individual parts as Laplace transforms of functions that we know.

3 Useful Techniques in Finding the Transforms

Several comments are important in practice and they are listed here:

- We can repeatedly use the derivative property to obtain the Laplace transforms of higher derivatives in terms of the Laplace transform of the original function, but an extra caution should be taken with regard to initial values;
- Partial fractions: because $1/(s-a), 1/(s^2+k^2)$ are so fundamental, it is helpful to use partial fractions to lead to an expression consisting of these components;
- Translation:

- if we see something familiar, but with a shifted variable $s - a$ instead of s involved, in the inverse we will just compensate by multiplying a factor e^{at};
- if we see something familiar, except multiplied by a factor e^{-as}, in the inverse we will compensate by shifting t to $t-a$, and multiplying a Heaviside function that has the effect of turn-on at $t = a$.

- Convolution: it tells us the inverse of $F(s)G(s)$ in terms of the inverses of F and G;
- Derivative and integral of F: the inverse of $F'(s)$ can be related to the inverse of F, by multiplying a factor of $-t$, and the inverse of $\int_s^\infty F d\sigma$ is therefore the inverse of F divided by t;
- Periodic functions: the transform formula can be shortened by integrating over one period, but a compensation must be made in terms of a factor $1/(a - e^{-ps})$.

Finally we introduced the Dirac delta function $\delta_a(t)$ in this chapter, as the function that is the inverse Laplace transform of $e{-}as$. The significance of this delta function is far beyond this fact, and we should keep in mind the physical interpretation behind the formalism.