Chapter 6
Nonlinear Systems and Phenomena

6.1 Stability and the Phase Plane

- We now move to nonlinear systems

- Begin with the first-order system for $\mathbf{x}(t)$

$$\frac{d}{dt}\mathbf{x} = \mathbf{f}(\mathbf{x}, t), \quad \mathbf{x}(0) = \mathbf{x}_0$$

- In particular, consider a 2x2 system

$$\frac{dx}{dt} = F(x, y)$$
$$\frac{dy}{dt} = G(x, y)$$

- Notice that the right-hand-sides have no explicit time dependence, this is so-called an **autonomous system**.

Solution displayed as a trajectory

- A solution to this system is of the form (x(t), y(t));
- Imagine (x(t), y(t)) to be the **position** coordinate of a particle at time t
- We can follow the particle on the plane along a **trajectory**, which represents a solution curve, and (x'(t),y'(t)) is the velocity of the particle at t
- Example: x(t) = cos(t), y(t) = sin(t) satisfies x'=-y, y'=x, and the trajectories are circles
- Question: how do x(t) and y(t) behave in t? More specifically, what happens when t approaches infinity?
- Of particular interest: when is the velocity equal to zero?
- Critical points: the solution (x*,y*) to

$$F(x, y) = 0$$
$$G(x, y) = 0$$

Examples of Critical Points

- Solve for a system of equations:

$$\frac{dx}{dt} = x - 2y + 3 = 0$$
$$\frac{dy}{dt} = x - y + 2 = 0$$

$$x = -1, \ y = 1$$

- Example 1 on page 372 (populations of rabbits and squirrels)

$$14x - 2x^2 - xy = 0$$
$$16y - 2y^2 - xy = 0$$

- Four solutions: (0,0), (0,8), (7,0), (4,6), discuss the relevances
- Notice that not all critical points have the same nature: some have trajectories going into the critical point, and some have them leaving the critical point

Phase Portraits

- Visualization of the solutions - help us to understand the system behind the equations
- Phase portrait: a picture that shows the **critical points** and **a collection of typical solution curves** or **trajectories** in the x-y plane;
- Slope field: **line segments** with slope dy/dx = G/F
- Direction field: typical **vectors** (x',y')
- Watch out for the tendency of the solution curves as t becomes large
- Example: direction field for the previous system

Critical Point Behavior

- Suppose (x*,y*) is a critical point;
- If we start from (x_0, y_0) nearby, does the solution curve lead to the critical point, or recede from it? This is crucial to our understanding of the nature of the critical point
- Example:

$$\frac{dx}{dt} = -x \qquad x(t) = x_0 e^{-t},\ \ y(t) = y_0 e^{-kt}$$
$$\frac{dy}{dt} = -ky$$

- Discuss the cases (i) k>0 (0<k<1, k=1, and k>1); (ii) k<0
- A critical point is called a **node** if
  - either every trajectory approaches it, or recede from it, as t approaches inf, and
  - every trajectory is tangent at (x*,y*) to some straight line
- **proper node**: no two different pairs of trajectories go in the same direction
- **sink** vs. **source**

Stability of Critical Points

- Intuitive question: if the particle gets close enough to the critical point, does it stay close?
- Rigorous definition: (x*,y*) is **stable,** if for each epsilon > 0, there exists a delta > 0, such that for all t

$$|\mathbf{x}(t) - \mathbf{x}_*| < \epsilon, \quad \text{if } |\mathbf{x}_0 - \mathbf{x}_*| < \delta$$

- A critical point is **unstable** if it is not stable
- Example: a damped pendulum, and an undamped pendulum
- Further definitions:
    - stable center: solution curves surround the critical point once they are close enough, typically with periodic solutions;
    - asymptotically stable: solutions begin sufficiently close to it, and approach it as t approaches inf;
    - spiral sink (belongs to asymptotically stable case) vs. spiral source (unstable)
- Example: x" + 2x' + 2x = 0

Categories of Stability Type

- Stable critical point:
    - asymptotically stable
        - proper or improper sink (node)
        - spiral sink
    - stable center
- Unstable critical point:
    - saddle point
    - proper or improper source (node)
    - spiral source

Four possibilities for a trajectory

- Qualitative nature of a solution near a critical point falls into one of the following:
  - (x(t),y(t)) approaches the critical point as t approaches infinity;
  - (x(t),y(t)) is unbounded with increasing t;
  - (x(t),y(t)) is a periodic solution with a closed trajectory;
  - (x(t),y(t)) spirals toward a closed trajectory as t approaches infinity.

6.2 Linear and Almost Linear Systems

- Consider the autonomous system $\frac{dx}{dt} = f(x, y), \quad \frac{dy}{dt} = g(x, y)$
- Linear system: both f and g have the form $ax + by + c$
- Advantage: the only critical point can be analyzed through an eigenvalue problem
- How can we use linear systems to help us understand nonlinear systems?
- Observe: **locally** the nonlinear system could be *approximated* by a linear system quite effectively
- Theoretical support: Taylor's expansion

$$F(x, y) = F(x_*, y_*) + F_x(x - x_*) + F_y(y - y_*) + \text{smaller terms}$$

- Change of coordinates so we can deal with a homogeneous problem: u=x-x*, v=y-y*

Linearization Near a Critical Point

- In variables u and v,

$$f(x_0+u, y_0+v) = f(x_0,y_0) + f_x(x_0,y_0)u + f_y(x_0,y_0)v + r(u,v)$$

- and a similar expression for g, where

$$\lim_{(u,v)\to 0} \frac{r(u,v)}{\sqrt{u^2+v^2}} = 0$$

- Assuming (x_0,y_0) is a critical point, the system for u and v is

$$\begin{aligned} \frac{du}{dt} &= f_x(x_0,y_0)u + f_y(x_0,y_0)v + r(u,v) \\ \frac{dv}{dt} &= g_x(x_0,y_0)u + g_y(x_0,y_0)v + s(u,v) \end{aligned}$$

- For small u and v ((x,y) sufficiently near (x_0,y_0)), if we drop r and s, we have the **linearized system**

$$\begin{aligned} \frac{du}{dt} &= f_x(x_0,y_0)u + f_y(x_0,y_0)v \\ \frac{dv}{dt} &= g_x(x_0,y_0)u + g_y(x_0,y_0)v \end{aligned}$$

$$\mathbf{u}' = \mathbf{J}\mathbf{u}$$

- J is the Jacobian matrix

Analysis of the Critical Point through a Linearization: Linear Analysis

- The linearized system leads us back to

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Critical point is (0,0), if the determinant of the matrix is nonzero, $ad - bc \neq 0$
- Interested in the behavior of solution near (0,0)
- Eigenvalue analysis again, need to discuss following cases for the eigenvalues
  - real and unequal, with the same sign;
  - real and unequal, with opposite signs;
  - real and equal;
  - complex conjugates with nonzero real part; or
  - pure imaginary.

Case 1: Real and Unequal with the Same Sign

- Two linearly independent solutions easily found;
- Solution has the form $\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t}$, one of two dominates
- Solution curves go to (0,0) along the dominating vector, so the node must be **improper**
- If the eigenvalues are positive, the critical point is a nodal **source**;
- If the eigenvalues are negative, the critical point is a nodal **sink.**
- Example

$$A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \qquad \lambda_1 = 2,\ \lambda_2 = 3$$

- Here the critical point is a nodal source since both eigenvalues are real and positive, and the solution curves are leaving along the eigenvector [1,1] corresponding to eigenvalue 3

Other Real Eigenvalue Cases

- Case 2: Unequal real with opposite signs:
  - Similar to the real and unequal with the same sign, except $\lambda_2 < 0 < \lambda_1$
  - Two asymptotes, along the eigenvectors;
  - One goes to zero, and the other goes to inf, so it is called a saddle point
- Case 3: Equal real:
  - May or may not need to find a generalized eigenvector:
    - Two linearly independent e-vectors: trajectories lie on straight lines - proper node
    - Only one independent e-vector: trajectories going along this eigenvector - improper node
  - Example: $A = \frac{1}{8}\begin{bmatrix} -11 & 9 \\ -1 & -5 \end{bmatrix}$
  - Both eigenvalues are -1, and there is only one linearly independent eigenvector, so trajectories are going into (0,0) along one direction [3,1]

Complex Conjugate Eigenvalues

- Because of the imaginary part, there is a “circle” feature
- Real part is crucial
- Nonzero real part: **spiral point**
  - negative real part: **spiral sink** (stable)
  - positive real part: **spiral source** (unstable)
- Zero real part - pure imaginary: it is a periodic solution - **spiral center**
- Example: $A = \begin{bmatrix} 4 & 5 \\ -5 & 3 \end{bmatrix}$
- Real part is positive so the critical point is a spiral source

Stability of Linear Systems

- We can categorize linear systems based on their eigenvalues, mostly the real parts
- **Theorem 1**
  - **assuming two eigenvalues for the matrix A, from x'=Ax, A nonsingular**
  - **the critical point is**
    - **asymptotically stable if both real parts are negative**
    - **stable but not asymptotically stable if the real parts are both zero**
    - **unstable if at least one real part is positive**
- Why is the categorization so important?
  - behavior of the linear system solutions
  - extend to the behavior of perturbed linear systems
  - extend to the behavior of almost linear systems
- Coefficients are never exactly measured, so we must consider perturbed systems

Stability of Perturbed Systems

- Perturbed system: a system where coefficients a, b, c, d are slightly altered
- Why bother? Coefficients in reality may not be exactly measured, so we should allow some room: the system we are given may be a little off from the system we need to consider
- Does the stability nature of a critical point get carried over after small perturbation in coefficients? Another way to ask: If we modify the system (in coefficients) slightly, do we change the stability type of a critical point?
- The answer is: it depends (on the eigenvalues)!
- The intermediate cases (zero real part, or repeated eigenvalues) will be the focus, as other cases are straightforward.
- It also extends to almost linear systems

Extension of Stability Type

- Instead of perturbations of the coefficients, what if we add some small nonlinear terms to the equations (adding r(x,y) and s(x,y))? This connects to the nonlinear problem we want to consider
- These are so called almost linear systems
- **Theorem 2**
  - **consider an almost linear system and a particular critical point of it**
  - **assuming two eigenvalues from the linearized system (around this critical point) have been found**
  - **the stability type of the critical point is one of the following:**
    - **for repeated real eigenvalues , the critical point is either a node or a spiral point, asymptotically stable or unstable depending on the sign of the eigenvalue;**
    - **for pure imaginary eigenvalues , the critical point is a center or a spiral point;**
    - **otherwise the type is the same as that of the associated linear system**
- Pay more attention to the borderline (in green above) cases
- Detailed categories summarized on page 393