Summary of Chapter 5

This chapter relies heavily on the linear algebra tools to study linear ODE systems, and we begin to move from just solving the equations to more qualitative but fundamental analysis of the systems. Crucial to all the work in this chapter is the concept of eigenvalues and eigenvectors of square matrices, and you are advised to refresh your familiarity with linear algebra materials if they appear to be rusty.

1 Important Linear Algebra Concepts

- Matrix and vectors;
- Matrix multiplication, unlike scalar multiplication, the order matters in a matrix-matrix, matrix-vector multiplication;
- Inner product between two vectors;
- Inverse matrix, diagonal matrix, determinant, nonsingular vs. singular matrices, and matrix functions;
- Linear independence of vectors;
- Eigenvalues and eigenvectors of a square matrix.

The last item will become the most important tool in dealing with the subject in this chapter.

2 The Eigenvalue Method

The linear system we want to solve is

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}$$

where $\mathbf{A}$ is a constant matrix (for our discussion in this chapter), and $\mathbf{f}$ is a vector of given functions of t. If $\mathbf{f} = 0$ the system is called homogeneous, and we have learned from previous chapters that homogeneous systems occupy the major work so we also discuss them first here.

The idea of using eigenvalues-eigenvectors came from the effort to extend the exponential solution to systems, and it is found that an eigenvalue problem needs to be solve in order to do that. The fact is that

$$\mathbf{x} = \mathbf{v}e^{\lambda t}$$

will be a solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ if $(\lambda, \mathbf{v})$ is an eigenvalue-eigenvector pair of the matrix $\mathbf{A}$. To find a general solution for the system, we need n linearly independent solutions, which will be supplied if we have n linearly independent eigenvectors. Unfortunately this is not always guaranteed for any $\mathbf{A}$, and our discussions will evolve around all different cases for eigenvalue-eigenvectors. As it turns out, the form of the solution depends crucially on the nature of eigenvalues and eigenvectors, and we will list them in the following.

One way to categorize is ask if n linearly independent eigenvectors exist for the matrix $\mathbf{A}$. If they do, a general solution can be readily written as

$$\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} + \ldots + c_n\mathbf{v}_n e^{\lambda_n t}.$$

If there are not n linearly independent eigenvectors (a fact pertaining to $\mathbf{A}$ that we can do nothing about it), a more elaborate form of solutions exist which requires the introduction of generalized eigenvectors.

The other way to distinguish different types of solution behavior is to inspect the eigenvalues and classify them accordingly. This is more relevant in the stability analysis where we are more concerned with the trend of the solution as t becomes large. All the cases have been summarized in detailed tables, and it is fair to say that the single most important sign to watch for is the real part of the eigenvalue. Here is a quick summary of cases:

1. Distinct real eigenvalues: independent eigenvectors can be found, each corresponding to one eigenvalue;

2. Complex eigenvalues: since they come in pairs and we can always find two linearly independent eigenvectors for a pair, we also have a set of linearly independent eigenvectors for this pair of eigenvalues;

3. Repeated eigenvalues: this is the most complicated case, and it all depends on if we can find a set of linearly independent eigenvectors with the number matching the multiplicity (thus the term defective in case the answer is no). If there is a defect, we will need to work on those generalized eigenvectors, which can be tedious, and it suffices to memorize only the basic situation.

3 Fundamental Matrix and Matrix Exponential

There is a neat way to represent the solutions to $\mathbf{x}' = \mathbf{A}\mathbf{x}$: a fundamental matrix $\mathbf{\Phi}$ simply combines n linearly independent solutions as column vectors together in a matrix, and it then satisfies

$$\mathbf{\Phi}' = \mathbf{A}\mathbf{\Phi}.$$

Then any solution can be written as $\mathbf{x} = \mathbf{\Phi}\mathbf{c}$ for some constant vector $\mathbf{c}$, and the IVP solution can be written as $\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}(0)^{-1}\mathbf{x}_0$.

Notice that fundamental matrices are not unique: different people can obtain different $\mathbf{\Phi}$, but if you compute $\mathbf{\Phi}(t)\mathbf{\Phi}(0)^{-1}$, everyone should get the same result. As it turns out, this matrix is actually the matrix exponential $e^{\mathbf{A}t}$, which is formally defined through a Taylor's series in $\mathbf{A}$. This suggests that will be two alternatives to solve a linear system: either obtaining the set of eigenvalues and eigenvectors of $\mathbf{A}$, or evaluate the matrix exponential. The matrix exponential seems to be a short cut, unfortunately for most matrices the series does not terminate easily, although there are instances that tricks can be used to facilitate the calculations.

4 Nonhomogeneous Systems

Finally we come to the general problem where there is a nonzero forcing term in the system. The method of undetermined coefficients can still be used, even though quite tedious; but we also introduced a method of variation of parameters, which extends well into other problems. The idea is based on the extension of representation of a solution $\mathbf{x} = \mathbf{\Phi c}$ for homogeneous solutions, to $\mathbf{x} = \mathbf{\Phi u}(t)$, and we face a new problem for $\mathbf{u}$ if we want this to be a specific solution:

$$\mathbf{u}' = \mathbf{\Phi^{-1} f}$$

which can be integrated if a fundamental matrix is available. The power of this approach is in this general formula, as the method of undetermined coefficients is based on trial-and-error.

5 Applications

The mass-spring system deserves a individual look because it is an example of turning a second-order system into a first-order system, and complex eigenvalues. In particular, if no damping is present, the eigenvalues are all purely imaginary, and we have a periodic solution illustrated in Theorem 1 on page 321. Here we see the importance of natural frequency, and find an interpretation for $\mathbf{x}_c$ (transient) and $\mathbf{x}_p$ (steady periodic) when a periodic forcing is applied.