Summary of Chapter 4

This chapter introduced the concept of system of differential equations. Besides the obvious applications in studying a system involving several unknown functions at the same time, systems of equations also cover an important extension: a higher-order equation can be recast as a system of first-order equations. Because of this generalization, for system of ordinary different equations we usually assume it is only first-order.

1 Systems Arising from Various Applications

It is a natural to consider systems of differential equations involving several unknown functions depending on the same variable, such as time t. In a typical application, a number of variables determine the system, and we can paint a picture by relating the variables and their derivatives based on some first principles. The examples we consider in this chapter include mass spring systems, where Newton's law is applied to establish conditions that lead to differential equations, and tank flow rate problems where a flux balance condition is used to derive differential equations.

If each equation involves only one unknown and it can be solved independent of others, then the system is no more than a repeated effort to deal with single equations. Complication arises when several unknowns are involved in the same equation, and these situations will certainly occur in most of the applications. The term "coupling" specifically refers to this situation and most of efforts in solving systems of equations are directed at this particular challenge. Obviously the more unknowns are present in each equation, the more difficult the system will become.

2 First-Order Systems

The first important observation is that any system, first order or higher-order, can be recast in terms of a first-order system, albeit at a cost of expanded system. To begin with, we notice that a single nth-order equation

$$x^{(n)} = f(t, x, x', x'', \dots, x^{(n-1)})$$

can be recast as a system for $x_1 = x, x_2 = x', x_3 = x'', \dots, x_n = x^{(n-1)}$:

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= x_3, \\ &\cdot \\ &\cdot \\ x_n' &= f(t, x_1, x_2, \dots, x_n) \end{aligned}$$

This idea can be extended to turn a higher-order system into a first-order system. For example, a 2×2 second-order system can be turned into a 4×4 first-order system using this technique.

A useful tool to visualize solutions of a two-dimensional system is the phase plane portraint where solutions are represented by trajectories. This approach is closely related to the concept of direction field, and various computer softwares are available, such as one listed on our course webpage. It should be pointed out visualization is more difficult for three-dimensional systems, and you will need imagination for even higher dimensional problems.

3 Linear Systems

In a linear system, each equation states that the derivative of one particular unknown is a *linear combination* of the other unknowns. This is the system for which we have systematic approaches to derive closed-form solutions, and they also serve the basis for studying more general nonlinear systems. There is a existence and uniqueness theorem for linear systems, which states the conditions that guarantee the exisitence and uniqueness of the solution to a linear system, with initial conditions for all the unknowns involved. As we expected, the conditions are just that all known functions involved, including the coefficients and forcing terms, should be continuous functions of t over the interval of interest.

4 Method of Elimination

Just like solving systems of linear equations, one can attempt to manipulate the equations so that one equation with one unknown emerges, and it can be solved alone. This process can be repeated one at a time until all the unknowns are solved. This is the idea behind this method of elimination, but we will only work on 2×2 systems here. For this approach we need to work with the concept of differential operators that are obtained through a polynomial of basic differential operators:

$$L = a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0$$

where D is the differential operator such that $Dx = x'$.

A 2×2 system can be written as

$$\begin{aligned} L_1 x + L_2 y &= f_1(t), \\ L_3 x + L_4 y &= f_2(t). \end{aligned}$$

The elimination process is similar to the one for linear systems, and we can **formally** write the equations for x and y as

$$\begin{vmatrix} L_1 & L_2 \\ L3 & L4 \end{vmatrix} x = \begin{vmatrix} f_1(t) & L_2 \\ f_2(t) & L4 \end{vmatrix},$$
$$\begin{vmatrix} L_1 & L_2 \\ L3 & L4 \end{vmatrix} y = \begin{vmatrix} L_1 & f_1(t) \\ L_3 & f_2(t) \end{vmatrix}.$$

We should note that each of them is a differential equation, as $L's$ are differential operators, but one involves only one unknown, therefore can be solved by the techniques we learned in previous chapters.

5 Numerical Methods

One advantage in using the vector-matrix notations is that all the numerical methods we discussed in Chapter 2 can be directly extended, simply by replacing the scalar variables with vector variables.