Chapter Three Summary

This chapter can be summarized as a detailed discussion on n-th order equation with constant coefficients. The equation we want to solve is

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x)$$

The discussion begins with general n-th order linear equations where these coefficients are continuous functions of x. However, when it comes to solving equations we limit ourselves to the case of constant coefficients.

1 General theory

First we learned to distinguish the case $f = 0$ and $f \neq 0$, where the former is called a homogeneous equation and the latter is called a nonhomogeneous equation. The homogeneous equation is very important even when the equation you are solving is non homogeneous, as its solution serves as a major part in the final solution for the nonhomogeneous equation. To be more specific, the road map to solve for a nonhomogeneous equation is the following:

1. Find a general solution to the homogeneous equation, which will contain n constants, and we call this y_c, the complementary function;
2. Find a particular solution y_p to the nonhomogeneous equation;
3. A general solution to the nonhomogeneous equation is therefore
$$y = y_c + y_p$$
4. If there are initial conditions, use n constants in y_c to satisfy all these n initial conditions. Note that these equations for determining the constants in y_c must be solved after y_p is introduced.

2 Homogeneous equation

One feature of the homogeneous equation is that $y = 0$ is always a solution, and a linear combination of solutions is also a solutions. The striking property of the homogeneous equation is that all solutions are covered by any set of n linearly independent solutions. In another word, if you have n linearly independent solutions, then you have all the possible solutions. It is therefore natural to focus on obtaining n linearly independent solutions.

Before we proceed, we need to examine the definition of linear independence of functions over an interval. The original definition is similar to the linear independence of n vectors in linear algebra. However, with n functions being solutions to a homogeneous differential equation, we can answer the question by calculating a determinant, called the Wronskian of n solutions $f_1, f_2, f_3, \ldots, f_n$. Whether these solutions are linearly independent or not, we just need to see if the Wronskian is zero or not. Note this criterion needs careful implementation in reality as a complicated calculation usually will not end up with the exact zero: we often have things called “nearly linearly independent”.

In the case of equations with constant coefficients, we have a systematic way to find these n linearly independent solutions. The revelation comes when we realize that by trying $y = e^{rx}$ we will land a solution if r satisfies the so-called characteristic equation:

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0$$

In theory this equation admits n solutions by the fundamental theorem of algebra. In practice we usually deal with only second, third, or fourth order equations. Regardless of the order, or the degree of the polynomial equation, there are three scenarios with the roots:

(a) distinct real;

(b) repeated real with multiplicity k;

(c) complex, in pairs.

For each of these cases respectively, we have solutions in the form

(a) ce^{rx};

(b) $\left(c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1}\right) e^{rx}$;

(c) $e^{ax}\left(c_1 \cos bx + c_2 \sin bx\right)$, where the roots are $r = a \pm bi$.

When we include all these solutions, counting multiplicity if necessary, we will arrive at a linear combination of n linearly independent solutions, with n constants $c_1, c_2, \ldots, c_n$. This is our general solution to the homogeneous equation.

3 Nonhomogeneous equation

For the nonhomogeneous equation, we will just need to find one particular solution, as simple as you can. Of course it depends on the right-hand-side f and we only consider certain kinds of functions for f: polynomial, exponential, and trigonometric.

It comes to distinguish many scenarios as what to try, and there is a lengthy list of guidelines. Instead of memorizing all these scenarios, we would like to emphasize the principle that we should start with something simple and obvious, when something failed we can increase the complexity by multiplying additional x factors to our guess. One thing we note is that once we include a function as part of the guess, its derivatives should also be included.

The procedure of undetermined coefficients start with the guess, a linear combination of these functions, and plug into the left-hand-side, compare with the right-hand-side, like term by like term, and arrive at a set of equations for the coefficients. You may start with the shortest equation, recover the easier ones first, and then move on to determine the others.

4 Initial conditions

Once we have the complementary function y_c and a particular solution y_p to the nonhomogeneous equation, we are ready to call $y = y_c + y_p$ our general solution to the nonhomogeneous equation. Only at this point we can determine the n coefficients in y_c (the coefficients in y_p have been determined in the previous step) by differentiating y and plug in the initial conditions. We expect to solve a $n \times n$ linear system to obtain these coefficients.

5 Application - mechanical vibration

In the applications we are no longer concerned with how to solve the equations, rather we focus on the interpretation of the solutions. In the mechanical vibration case, the equation is

$$mx'' + cx' + kx = F(t) = F_0 \cos \omega t$$

The homogeneous case $F = 0$ is called free oscillation, while it's called forced if $f \neq 0$. If $c = 0$ we call it an undamped oscillation. With combinations, we also have free damped, free undamped oscillation, and so on.

The most important property of this mechanical system is the circular frequency $\omega_0 = \sqrt{k/m}$. In the undamped case, what we need to watch out is if the frequency applied to the system ω is close to ω_0, in which case a resonance occurs and the system will suffer a major crash.

When there is damping, as always in nature, we need to determine if the damping is strong enough. The critical value is $c_{cr} = \sqrt{4mk}$ and we can divide the cases into (1) underdamped, (2) critically damped, and (3) overdamped. The solutions will be different, depending on the case.

6 Application - RCL circuit

This application can be viewed as an electric analogy of the mechanical vibration. It helps to compare the equation with corresponding parameters:

$$LI'' + RI' + \frac{1}{C}I = E_0\omega \sin \omega t$$

Then everything should follow, just like the vibration model.