Chapter Two Summary

In this chapter we discussed two models: the population model and acceleration-velocity model. We then introduced and analyzed the idea of the fundamental numerical method to solve IVPs.

1 Population Models

The target is the population of certain species at time $t : P(t)$. We model it through its rate of change with respect to time. Three factors are at play: birth, death, and harvest. Note that the birth rate is defined as the number of birth per unit time, **per unit of population**, and there is a similar definition for the death rate. The general equation can be written as

$$\frac{dP}{dt} = kP(M-P) - h$$

In the case $h = 0$, we can see two constant solutions $P = 0$ and $P = M$. Our main question is what happens to P as time goes on, if P starts below M, or above M. This kicks off our discussion on the stability of the solutions. Any value that makes the right-hand-side zero is called a critical point. We have ways to determine if a particular critical point is stable or not. One way is to look at how the right-hand-side changes sign across a critical point.

The solution to the above equation with $k > 0$ (logistic equation) is

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-kMt}}$$

and we can see that the approaching to M is exponential. If we have $P - M$ instead of $M - P$ in the equation, we can achieve the same effect by replacing k with $-k$ and the solution is

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{kMt}}$$

This, however, changes the nature quite a bit. Now for $P_0 > M$ at some finite t the denominator is actually zero and the species experiences the doomsday.

With $h > 0$, we should factor the right-hand-side to arrive at $k(N - P)(P - H)$, which can be viewed as a modification to the original logistic equation. Just think about $P_1 = P - H$, so the right-hand-side is $kP_1(M - P_1)$ with $M = N - H$.

The acceleration-velocity model is a direct application of Newton's Laws. Our task is to set up the equation that we can solve. If we use y to denote the vertical position (height) of certain object, then $v = y'$ is the velocity and $a = v' = y''$ is the acceleration. A crucial step in writing down the equation is to verify that the signs in front of the forces are consistent. We consider two forces: gravitational and resistant, where the first always points downward, and the second goes against the velocity.

2 Numerical Methods

Euler's method is the basis for every other numerical method, and it is based on the geometric interpretation of the equation itself. Any extension to improve can be viewed as either (1) to improve the accuracy of the slope at which the approximate solution curve advances; or (2) a more accurate way to integrate the right-hand-side.