Chapter One Summary

1 Categories

By name, differential equations are equations that involve unknown functions and their derivatives. In this semester we start with **ordinary** differential equations where only functions of one variable are considered. Later we will study **partial** differential equations where partial derivatives of the unknown functions are involved.

We can categorize equations according to several considerations:

1.1 Linear vs. nonlinear

In a linear equation, the unknown function and its derivatives appear linearly. That is, each of them is only multiplied by a known function of the independent variable (including constant), and all terms are put together as a sum, then set to another known term. Any equation that is not linear is called a nonlinear equation. Linear equations are much easier to solve and analyze than nonlinear equations. It is important to realize early on whether you are dealing with a linear or a nonlinear equation.

1.2 Order of the equation

It is simply the highest order of derivative appearing in the equation. Obviously low order equations are easier to deal with than higher order ones. We will show later that any high order equation can be transformed to a *system* of first-order equations.

1.3 Constant coefficient or variable coefficient

If you recognize the equation to be linear, the next question you can ask is if it is a equation with constant coefficients or not. This should be obvious to tell.

2 First-order Equations

First-order equations are supposed to be easiest, so we consider both linear and nonlinear cases. The general form of the equation is

$$\frac{dy}{dx} = f(x, y)$$

We discussed two ways to solve: direct integration, and a geometric approach based on slope field and solution curves.

The integration approach is not guaranteed to work, and we will need to manipulate the equation to find the trick. There are two situations that we know we can get it to work: (1) when $f(x, y)$ is a product of a function of x and another function of y; and (2) if the equation is linear.

In the case $f(x, y) = g(x)h(y)$, we set up the equation in the form

$$\frac{dy}{h(y)} = g(x)\,dx$$

and integrate both sides separately to arrive at an equation involving x and y, but not y'. The solution will be implicitly specified through that equation. Because there is an integration involved, a constant will turn up in the solution, and it can be fixed once an initial condition is supplied. If no initial condition is mentioned, we should leave a constant in the solution.

In the case when the equation is linear:

$$y' + P(x)y = Q(x)$$

we multiply a factor ρ in a hope that the left-hand-side is just $d(\rho y)$ so we can integrate to arrive at the solution

$$y(x) = \frac{1}{\rho(x)}\left[\int \rho(x)Q(x)\,dx + C\right], \quad \rho(x) = e^{\int P(x)\,dx}$$

Finally, we state the general existence/uniqueness theorem for first-order equations with initial conditions (so-called IVP). What we need to verify is the continuity of two functions: $f(x, y)$ and $D_y f(x, y)$, in a region containing the initial value (x_0, y_0). If the equation turns out to be linear, then we can say more about the domain where the solution exists.