Discriminating between level shifts and random walks: a delay time approach

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Abstract: We derive the limit distribution of a stopping time used to sequentially detect a change from stationary to random walk behavior. Results are compared to known results in a level shift setting and are underlined by a simulation study.

MSC 2000: 91B84, 62L10
Key words: sequential tests, CUSUM–type test procedures, delay time, Brownian motion, change–points, random walk

1 Introduction

In this paper, we will derive the limit distribution of a stopping rule, originally designed to detect on–line level shifts, in a different environment in which the underlying sequence changes from stationarity to random walk behavior at an unknown time–point.

Detecting changes in the mean—also known as level shifts—has long been a focus of attention in a variety of statistical subdisciplines ranging from quality control (see [8] for the first contribution), climatology (see, eg, [7]) and econometrics (see, eg, [9]). The major part of the literature body available is devoted to tests of one–shot type, that is, all observations have been collected before the statistical analysis commences. Recently, following the papers of Chu et al. [5] and Horváth et al. [6], sequential approaches, in particular for linear models, have received some greater attention. We only cite the contribution Aue et al. [2] and refer the interested reader to the references therein. Here, in contrast, a decision has to be made on–line as new data arrive steadily. We shall study a similar sequential setting for univariate random variables.

Our sequential test procedure is a stopping rule of the form $\tau = \inf\{k \geq 1 : \Gamma(k) \geq g(k)\}$, where $\Gamma(k)$ is the value of a suitable detector at lag k, and $g(k)$ the value of a corresponding threshold function at the same time–point. The procedures will be based conveniently on CUSUM–type detectors, the choice of the threshold function will follow both mathematical and practical reasons.

The paper is organized as follows. In Section 2, we introduce the model, state assumptions and discuss results obtained in Aue and Horváth [1], who derived the limit distribution of a stopping rule τ under the level shift alternative. The main aim, however, is to present new results extending the existing theory by deriving the asymptotic behavior of τ under the random walk alternative. Simulations providing

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Research partially supported by NATO grant PST.EAP.CLG 980599, NSF OTKA grant INT–0223262 and an NSF VIGRE grant.

the graph of the density of the limit distribution and some selected critical values can be found in Section 3. Eventually, proofs are given in Section 4.

2 Model assumptions and discussion of results

(a) Model assumptions. Motivated by new research in econometrics, which indicates cost–free data in a wide variety of applications (see Chu et al. [5]), we present a sequential test procedure that is aimed at detecting possible changes in the mean of the underlying process governing the random behavior. This setting is an adaptation of the linear model theory obtained in Horváth et al. [6]. More precisely, we study a sequence of random variables $\{X_k\}$ defined by

$$X_k = \mu_k + \varepsilon_k, \qquad k = 1, 2, \ldots, \tag{1}$$

under three different scenarios, in which the drift terms $\{\mu_k\}$ and the centered error terms $\{\varepsilon_k\}$ satisfy distinctively different conditions. But first, we need another assumption stating that the process remains in control in a so–called training period of size m. For our purposes this means,

$$\mu_1 = \ldots = \mu_m = \mu. \tag{2}$$

Assumption (2) is particulary important because the testing procedure to be defined in part (b) of the section can use the first m observations as base for comparisons with future values.

(I) Our first scenario describes the null hypothesis of structural stability. Here, the trend terms $\{\mu_k\}$ are constant over time, that is

$$\mathcal{H}_0 : \quad \mu_k = \mu, \qquad k = m+1, m+2, \ldots \tag{3}$$

The error sequence is assumed to satisfy the two conditions

$$\left| \sum_{i=1}^{m} \varepsilon_i \right| = \mathcal{O}_P\left(\sqrt{m}\right) \qquad (m \to \infty), \tag{4}$$

$$\sup_{1/m \le t \le \infty} \frac{1}{(mt)^{1/\nu}} \left| \sum_{i=m+1}^{m(1+t)} \varepsilon_i - \sigma W_m(mt) \right| = \mathcal{O}_P(1) \qquad (m \to \infty), \tag{5}$$

where $\{W_m(t) : t \geq 0\}$ denotes a sequence of Brownian motions, $\sigma > 0$ and $\nu > 2$.

(II) Under the level shift alternative, the drift terms $\{\mu_k\}$ satisfy

$$\mathcal{H}_A^{(1)} : \quad \mu_k = \begin{cases} \mu & k = m+1, \ldots, m+k^*-1, \\ \mu + \Delta_m & k = m+k^*, m+k^*+1, \ldots \end{cases} \tag{6}$$

where $k^* \geq 1$ is unknown, while the error terms are still assumed to fulfill conditions (4) and (5).

(III) Under the random walk alternative, it holds,

$$\mathcal{H}_A^{(2)}: \quad \mu_k = \mu, \qquad k = m+1, m+2, \ldots,$$

$$\frac{1}{\sqrt{n}} \varepsilon_{\lfloor nt \rfloor} \overset{\mathcal{D}[0,1]}{\longrightarrow} \sigma W(t) \qquad (n \to \infty), \tag{7}$$

where $\{W(t) : t \geq 0\}$ denotes Brownian motion, $\overset{\mathcal{D}[0,1]}{\longrightarrow}$ convergence in the Skorohod space $\mathcal{D}[0,1]$ and $\sigma > 0$. Here, the drift terms remain constant, but the error terms change from stationary behavior under $\mathcal{H}_0$ to a limiting random walk.

The assumptions on the error sequences are flexible enough to include large classes of dependent random processes in the setting. These include ARMA time series, GARCH–type processes and many more. For more details on possible choices of $\{\varepsilon_k\}$ confer [1].

(b) Test statistics. Our assumption that data come in cost–free allows us for a simplification of the typical Wald–type test statistics, which have to take into account that, even under the null hypothesis of no change, test procedures have to terminate. In contrast, we are satisfied with continuing the monitoring if no change has been indicated. The test statistic under consideration is based on the CUSUM detector

$$Q(m,k) = \sum_{i=m+1}^{m+k} X_i - \frac{k}{m} \sum_{i=1}^{m} X_i,$$

which compares the sample means after the training period, $\frac{1}{k} \sum_{i=m+1}^{m+k} X_i$, with those obtained from the training period, $\frac{1}{m} \sum_{i=1}^{m} X_i$. This can be seen easily by dividing the right–hand side of the latter display by k.

We stop and state a change if $Q(m,k)$ attains too large values, ie, exceeds the corresponding value of the threshold function

$$g_\gamma(m,k) = c(\alpha)\sqrt{m} \left(1 + \frac{k}{m}\right) \left(\frac{k}{m+k}\right)^\gamma, \qquad \gamma \in [0, 1/2)$$

for the first time. That is, the stopping rule is given by

$$\tau(m) = \min \{k \geq 1 : |Q(m,k)| \geq g_\gamma(m,k)\}.$$

Note that the constant $c(\alpha)$ in the definition of $g_\gamma(m,k)$ is chosen such that $P\{\tau(m) < \infty\} = \alpha$ under $\mathcal{H}_0$. More general choices of threshold functions are possible and have been discussed, eg, in [2, 3, 6]. However, besides mathematical tractability, the usage of $g_\gamma(m,k)$ offers moreover some flexibility to adjust the sensitivity of the testing procedure by choosing different values of the parameter γ.

See the discussion below for more insight.

(c) Limit theorems. Theorems 2.1 and 2.2 to be stated shortly are simple adaptations of the corresponding theorems for linear models in Horváth et al. [6]. They provide the limit distribution of the testing procedure under the null hypothesis of no change and the asymptotic consistency under a level shift.

Theorem 2.1 (Asymptotic under $\mathcal{H}_0$). *Let $\{X_i\}$ be a sequence of random variables satisfying (1)–(5). Then,*

$$\lim_{m\to\infty} P\left\{\frac{1}{\hat{\sigma}_m}\sup_{k\geq 1}\frac{|Q(m,k)|}{g_\gamma(m,k)}\leq 1\right\} = P\left\{\frac{|W(t)|}{t^\gamma}\leq c\right\},$$

where $c=c(\alpha)$, $\{W(t): t\in[0,1]\}$ denotes a Brownian motion and $\hat{\sigma}_m^2$ is a suitable variance estimator.

Theorem 2.2 (Asymptotic under $\mathcal{H}_A^{(1)}$). *Let $\{X_i\}$ be a sequence of random variables satisfying (1), (2) and (4)–(6). Then, under either of the following conditions,*

1. *$\Delta_m = \Delta$ is constant and $k^* = o(m)$ as $m\to\infty$,*
2. *$\Delta_m \to 0$ but $\sqrt{m}|\Delta_m|\to\infty$ and $k^* = \lfloor \beta m\rfloor$, $\beta > 1$ fixed, as $m\to\infty$, where $\lfloor\cdot\rfloor$ denotes integer part,*

it holds,

$$\frac{1}{\hat{\sigma}_m}\sup_{k\geq 1}\frac{|Q(m,k)|}{g_\gamma(m,k)}\xrightarrow{P}\infty \qquad (m\to\infty),$$

where $\hat{\sigma}_m^2$ is a suitable variance estimator.

With Theorem 2.1 and Theorem 2.2 the behavior under both $\mathcal{H}_0$ and $\mathcal{H}_A^{(1)}$ is completely characterized. However, it has to be mentioned that, unless $\gamma = 0$, the limit distribution appearing in Theorem 2.1 is unknown and one has to rely on simulations to obtain critical values to perform asymptotic tests.

Simulation studies in [6] also imply that it is advisable to use values of γ close to the (excluded) boundary value 1/2 if a change has to be detected quickly and if an increased false alarm rate is acceptable. A theoretical proof of this empirical fact is due to Aue and Horváth [1]. It requires the additional assumptions

$$\Delta_m \to 0 \qquad \text{but} \qquad \sqrt{m}|\Delta_m|\to\infty \qquad (m\to\infty) \tag{8}$$

and

$$k^* = \mathcal{O}\left(m^\theta\right) \qquad \text{with some} \quad 0\leq\theta<\left(\frac{1-2\gamma}{2(1-\gamma)}\right)^2. \tag{9}$$

In other words, the size of change Δ_m is moderate and the change does not occur too late in the sample.

Theorem 2.3 (Limit distribution of $\tau(m)$ under $\mathcal{H}_A^{(1)}$). *Let $\{X_i\}$ be a sequence of random variables satisfying (1), (2), (4)–(6), (8) and (9). Then, for all real x,*

$$\lim_{m\to\infty} P\left\{\frac{\tau(m)-a(m)}{b(m)} \le x\right\} = \Phi(x),$$

where Φ denotes the distribution function of a standard normal random variable,

$$a(m) = \left(\frac{cm^{1/2-\gamma}}{\Delta_m}\right)^{1/(1-\gamma)} \qquad \text{and} \qquad b(m) = \frac{\sqrt{a(m)}\sigma}{(1-\gamma)\Delta_m}.$$

It can be checked that $a(m)$ gets smaller with γ chosen closer to $1/2$. Since, in particular, $\tau(m) \approx a(m)$ if m is sufficiently large, this gives the theoretical justification of a faster change detection in the neighborhood of $1/2$.

Recent research indicates that many test statistics designed to detect level shifts are also sensitive to other types of alternatives such as the random walk setting. In what follows, we are going to derive the limit distribution of the stopping rule $\tau(m)$ under $\mathcal{H}_A^{(2)}$. We restrict the discussion to the special case of an immediate change $k^* = 1$ such that $\mu_k = 0$ for all $k \ge m+1$. More general results will be discussed elsewhere. We obtain the following theorem.

Theorem 2.4 (Limit distribution of $\tau(m)$ under $\mathcal{H}_A^{(2)}$). *Let $\{X_i\}$ be sequence of random variables satisfying (1), (2), (4) and (7) with $\mu_k = 0$. If $k^* = 1$, then,*

$$\lim_{m\to\infty} P\left\{\tau(m) \ge x m^{(1-2\gamma)/(3-2\gamma)}\right\} = P\left\{\max_{0\le t\le x} \frac{\sigma}{t^\gamma}\left|\int_0^t W(u)du\right| \le c\right\}, \qquad (10)$$

where $c = c(\alpha)$, $\{W(t) : t \ge 0\}$ denotes Brownian motion.

Simulations providing selected values of the stopping rule are given in the next section. The proof of Theorem 2.4 is relegated to Section 4.

3 A simulation study

In this section we are going to provide a small Monte Carlo simulation to obtain the distribution function of the limiting random variable in (10) for two choices of the sensitivity parameter, namely $\gamma = 0$ and $\gamma = 0.49$. To do so, we note that, after an application of the scale transformation for Brownian motions, we get that, for all $x > 0$,

$$\max_{0\le t\le x} \frac{1}{t^\gamma}\left|\int_0^t W(u)du\right| \overset{\mathcal{D}}{=} x^{3/2-\gamma} \max_{0\le t\le 1} \frac{1}{t^\gamma}\left|\int_0^t W(s)ds\right|,$$

where $\overset{\mathcal{D}}{=}$ stands for equality in distribution. Hence, equation (10) can be rewritten as

$$\lim_{m\to\infty} P\left\{\tau(m) \ge x m^{(1-2\gamma)/(3-2\gamma)}\right\} = F_\gamma\left(\frac{c}{\sigma} x^{\gamma-3/2}\right),$$

where

$$F_\gamma(u) = P\left\{\max_{0\le t\le 1} \frac{1}{t^\gamma}\left|\int_0^t W(s)ds\right| \le u\right\}.$$

Our simulations will provide the graphs of the distribution function F_γ for the choices $\gamma = 0$ and $\gamma = 0.49$.

Figure 1 exhibits the density functions of F_0 and $F_{0.49}$. The density of F_0 is skewed farther to the left than the density of $F_{0.49}$, which takes larger values than the one of F_0 after the latter density attains its maximum. Using Monte Carlo simulation, we computed z_α satisfying $F_\gamma(z_\alpha) = \alpha$ for the same two values of γ when $\alpha = 0.01, 0.05, 0.10, 0.90, 0.95$, and 0.99. These values are reported in the following table.

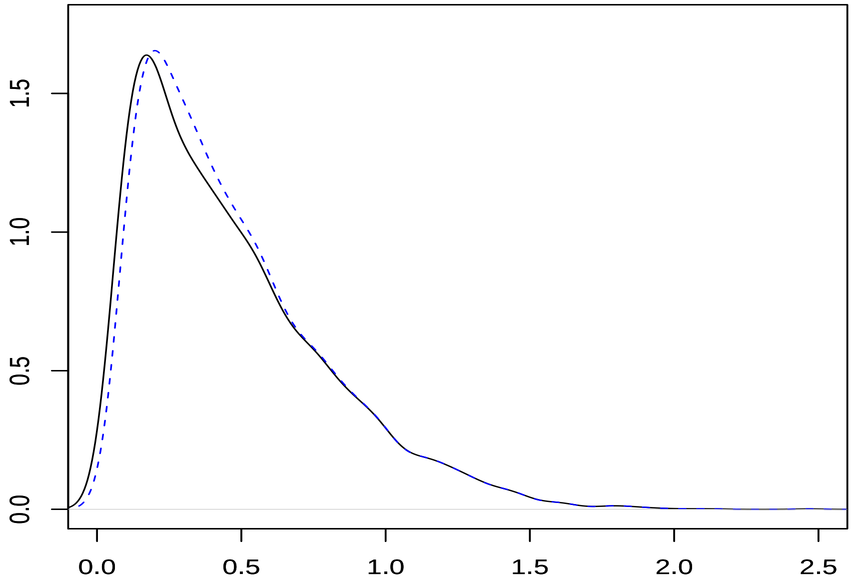

Figure 1: Densities of F_γ for $\gamma = 0$ (straight line) and $\gamma = 0.49$ (dashed line).

α	$\gamma = 0$	$\gamma = 0.49$
0.01	0.049	0.066
0.05	0.084	0.105
0.10	0.113	0.134
0.90	0.954	0.954
0.95	1.126	1.126
0.99	1.467	1.467

Table 1: Selected critical values for F_γ.

4 Proof of Theorem 2.4

Let $N = xm^{(1-2\gamma)/(3-2\gamma)}$. The proof of Theorem 2.4 is based on the fact that

$$P\{\tau(m) > N\} = P\left\{\max_{1\le k\le N}\frac{|Q(m,k)|}{g_\gamma(m,k)} \le c\right\}.$$

The first goal is, hence, to replace $Q(m,k)$ by a simpler expression.

Lemma 4.1. *If the assumptions of Theorem 2.4 are satisfied, then,*

$$\max_{1\le k\le N}\frac{1}{g_\gamma(m,k)}\left|Q(m,k) - \sigma\int_1^k W(u)du\right| = o_P(1) \qquad (m\to\infty),$$

Proof. Since $\mu_k = 0$ for all $k \ge m+1$ by assumption, it follows from (7) that

$$\max_{1\le k\le N}\frac{1}{g_\gamma(m,k)}\left|\sum_{i=m+1}^{m+k} X_i - \sigma\int_1^k W(u)du\right| = o_P(1) \qquad (m\to\infty).$$

Moreover, since $|\sum_{i=1}^m X_i| = \mathcal{O}(m)$ by the strong law of large numbers,

$$\max_{1\le k\le N}\frac{1}{g_\gamma(m,k)}\frac{k}{m}\left|\sum_{i=1}^m X_i\right| = \mathcal{O}_P\left(\frac{N^{1-\gamma}}{m^{1/2-\gamma}}\right) = o_P(1) \qquad (m\to\infty)$$

and the proof of Lemma 4.1 is complete. □

The second lemma simplifies the integral expression obtained in Lemma 4.1.

Lemma 4.2. *If the assumptions of Theorem 2.4 are satisfied, then, as $m\to\infty$,*

$$\max_{1\le k\le N}\frac{c}{g_\gamma(m,k)}\left|\int_1^k W(u)du\right| \overset{\mathcal{D}}{=} \max_{0\le k/N\le x}\left(\frac{k}{N}\right)^{-\gamma}\left|\int_{1/N}^{k/N} W(y)dy\right| + o_P(1)$$

Proof. Write

$$g_\gamma(m,k) = cm^{1/2}\left(1+\frac{k}{m}\right)^{1-\gamma}\left(\frac{k}{m}\right)^{\gamma}$$

and note that $1 + k/m \to 1$ for all $k = 1, \ldots, N$. Hence, the assertion of Lemma 4.2 follows from Lemma 4.1 after a change of variables, letting $y = u/N$, and an application of the scale transformation for Brownian motions. □

Theorem 2.4 is now an immediate consequence of Lemmas 4.1 and 4.2 combined with the almost sure continuity of Brownian motions.

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