KAZHDAN-LUSZTIG ALGORITHM FOR WHITTAKER MODULES WITH ARBITRARY INFINITESIMAL CHARACTERS

by

Qixian Zhao

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STATEMENT OF DISSERTATION APPROVAL

The dissertation of **Qixian Zhao**

has been approved by the following supervisory committee members:

Dragan Miličić,	Chair(s)	**04 February 2023** Date Approved
Peter E. Trapa,	Member	**03 February 2023** Date Approved
Henryk Hecht,	Member	**05 February 2023** Date Approved
Karl E. Schwede,	Member	**03 February 2023** Date Approved
Leticia Barchini,	Member	**17 March 2023** Date Approved

by **Davar Khoshnevisan**, Chair/Dean of

the Department/College/School of **Mathematics**

and by **David B. Kieda**, Dean of The Graduate School.

Abstract

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. We prove a Kazhdan-Lusztig algorithm for Whittaker modules of $\mathfrak{g}$ with arbitrary infinitesimal characters. This leads to a description of the block decomposition of the category of Whittaker modules with non-integral infinitesimal characters, and also to a character formula for irreducible Whittaker modules, generalizing previous work of Miličić-Soergel and Romanov for integral infinitesimal characters.

Contents

List of Figures

List of Tables

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Chapter 1

Introduction

This dissertation studies Whittaker modules of a complex semisimple Lie algebra $\mathfrak{g}$. To motivate this subject, let G_0 be a suitably nice real semisimple Lie group. In order to better study properties of a smooth representation V of G_0, one natural question is whether it can be realized as a subspace of the space of smooth functions on G_0, analogous to realizing finite dimensional vector spaces over $\mathbb{R}$ as $\mathbb{R}^n$. Such a realization can be obtained via *matrix coefficients*. These are functions c_{ν,ν^*} : $G_0 \to \mathbb{C}$, $g \mapsto \langle \nu^*, g \cdot \nu \rangle$ depending on a choice of $\nu \in V$ and $\nu^* \in V^*$. If ν^* is fixed, then (under suitable assumptions) $\nu \mapsto c_{\nu,\nu^*}$ defines a map $V \to C^\infty(G_0)$ whose image lies in the representation generated by the c_{ν,ν^*}'s.

One tractable situation is when both ν and ν^* transform according to some finite-dimensional representations of a maximal compact subgroup K_0 of G_0. This is called the K-finite case, and in this case c_{ν,ν^*} is determined by its restriction to a certain abelian subgroup $A_0 \cong (\mathbb{R}^+)^n$ of G_0, namely the split torus appearing in the Iwasawa decomposition $G_0 = K_0 A_0 N_0$. If $G_0 = \mathbf{SL}(2, \mathbb{R})$, then $A_0 \cong \mathbb{R}^+$ and $c_{\nu,\nu^*}|_{A_0}$ satisfies a Legendre equation and is a hypergeometric function. On the other hand, if ν^* no longer satisfies the above K-finite condition but transforms instead according to a one dimensional representation of the unipotent subgroup N_0 in the Iwasawa decomposition, then c_{ν,ν^*} is still determined by $c_{\nu,\nu^*}|_{A_0}$. For $G_0 = \mathbf{SL}(2, \mathbb{R})$, $c_{\nu,\nu^*}|_{A_0}$ instead satisfies the Whittaker equation. In this case, ν^* is called a Whittaker functional on V, and the induced map $V \to C^\infty(G)$ is called a Whittaker model of V. They were first considered by Jacquet [Jac67], and has found fruitful applications in many different settings. There have been many interesting papers about them, for example Shalika [Sha74], Casselman-Hecht-Miličić [CHM00], etc.

We are interested in the same concept for representations of Lie algebras. For a complex semisimple Lie algebra $\mathfrak{g}$, a maximal nilpotent subalgebra $\mathfrak{n}$, and a character $\eta : \mathfrak{n} \to \mathbb{C}$, we consider *Whittaker modules* which are representations V of $\mathfrak{g}$ generated by a vector $\nu \in V$ on which $\mathfrak{n}$ acts by the character η. Kostant studied these modules in his beautiful paper [Kos78]. He showed that in the non-degenerate case (when η is nonzero on all simple root spaces of $\mathfrak{n}$), the category of Whittaker modules has a very simple description.

Problem. *Find a composition series and the multiplicities of composition factors of Whittaker modules.*

Algebraic treatments on this problem were successful. As is already mentioned, Kostant treated the non-degenerate case. In the degenerate case, McDowell constructed and studied *standard Whittaker modules*, which are analogs of Verma modules [McD85]. Based on McDowell's work, Miličić and Soergel later gave a precise answer to the problem using algebraic arguments for modules with integral infinitesimal characters [MS97]. Here the *infinitesimal character* describes the action of the center of the enveloping algebra $\mathcal{U}(\mathfrak{g})$. Integrality is a usual assumption and is the "basic case" compared to general infinitesimal characters.

It was observed by Miličić and Soergel that the problem has a solution based on the localization theory of Beilinson and Bernstein, similar to the solution of Kazhdan-Lusztig conjecture for Verma modules. A geometric proof of Kostant's result in the non-degenerate case was obtained in 1986 but was published much later [MS14]. The general argument for Verma modules was not translated to Whittaker modules until a key ingredient was proven by Mochizuki [Moc11], namely the decomposition theorem for general holonomic $\mathcal{D}$-modules. Based on this, Romanov proved an algorithm for computing multiplicities of Whittaker modules with integral infinitesimal characters in her dissertation [Rom21].

Goal. *Generalize Miličić-Soergel's and Romanov's results to arbitrary infinitesimal characters.*

In the remaining part of this introduction, we give a brief presentation on the preliminaries on Whittaker modules and Beilinson-Bernstein's localization theorems, state the main results, and explain the idea of our proof. The introduction will conclude with an outline of the dissertation.

1.1 Preliminaries on Whittaker modules

In this section we present relevant facts on Whittaker modules without proof. References include [Kos78], [McD85], [MS97], [Mil], and [Rom21].

Let us start with notations. Let $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{n}$, $\mathfrak{h}$ be a complex semisimple Lie algebra, a Borel subalgebra, the nilpotent radical of the Borel, and a Cartan subalgebra. Let $G \supset B \supset N$, H denote a complex connected algebraic group with Lie algebra $\mathfrak{g}$, and subgroups corresponding to $\mathfrak{b}, \mathfrak{n}, \mathfrak{h}$ respectively. Write Σ for the root system of $(\mathfrak{g}, \mathfrak{h})$, $\Pi \subset \Sigma^+ \subset \Sigma$ for the set of simple and positive roots determined by $\mathfrak{b}$, ρ for the half sum of roots in Σ^+, and W for the Weyl group of Σ. For any

Lie algebra character $\eta : \mathfrak{n} \to \mathbb{C}$, we define a subset Θ of simple roots by

$$\Theta = \{\alpha \in \Pi \mid \eta \text{ is nonzero on the } \alpha\text{-root space in } \mathfrak{n}\}. \tag{1.1.1}$$

We then let a subscript Θ denote subobjects defined by Θ. Thus Σ_Θ is the root subsystem of Σ generated by roots in Θ, $\Sigma^+_\Theta = \Sigma^+ \cap \Sigma_\Theta$, ρ_Θ is the half sum of roots in Σ^+_Θ, and W_Θ is the subgroup of W generated by reflections of roots in Θ. Finally, let $\mathcal{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$, and let $\mathcal{Z}(\mathfrak{g})$ be the center of $\mathcal{U}(\mathfrak{g})$.

By an *infinitesimal character*, we mean a $\mathbb{C}$-algebra homomorphism from $\mathcal{Z}(\mathfrak{g})$ to $\mathbb{C}$. Via the Harish-Chandra isomorphism $\mathcal{Z}(\mathfrak{g}) \cong \mathrm{Sym}(\mathfrak{h})^W$ (this is the map $\gamma \circ \varphi|_{\mathcal{Z}(\mathfrak{g})}$ in [Dix96, Theorem 7.4.5]), infinitesimal characters χ_θ are parameterized by W-orbits θ in $\mathfrak{h}^*$, where any $\lambda \in \theta$ determines χ_θ by the composition

$$\mathcal{Z}(\mathfrak{g}) \cong \mathrm{Sym}(\mathfrak{h})^W \hookrightarrow \mathrm{Sym}(\mathfrak{h}) \xrightarrow{\lambda} \mathbb{C}.$$

Consequently each θ determines a maximal ideal $\ker \chi_\theta$ in $\mathcal{Z}(\mathfrak{g})$. We let $\mathcal{U}_\theta = \mathcal{U}(\mathfrak{g})_\theta = \mathcal{U}(\mathfrak{g})_\lambda$ denote the quotient $\mathcal{U}(\mathfrak{g})/\mathcal{U}(\mathfrak{g}) \ker \chi_\theta$. For us, θ will always denote the W-orbit of λ.

The category of **Whittaker modules**, denoted by $\mathcal{N}$, is the full subcategory of all $\mathfrak{g}$-modules consisting of those that are finitely generated over $\mathfrak{g}$, locally finite over $\mathfrak{n}$, and locally finite over $\mathcal{Z}(\mathfrak{g})$. Here we say a module over an algebra is locally finite if every element generates a finite dimensional subspace. $\mathcal{N}$ has a full subcategory $\mathcal{N}_\theta$ consisting of modules on which $\mathcal{Z}(\mathfrak{g})$ acts by χ_θ. Similarly, $\mathcal{N}_\eta$ consists of modules on which $\xi - \eta(\xi)$ acts locally nilpotently for all $\xi \in \mathfrak{n}$. We set $\mathcal{N}_{\theta,\eta} = \mathcal{N}_\theta \cap \mathcal{N}_\eta$. By McDowell's work [McD85], every object in $\mathcal{N}$ has finite length, and each irreducible object is contained in one of the $\mathcal{N}_{\theta,\eta}$'s. Different η's define different subcategories $\mathcal{N}_{\theta,\eta}$'s of $\mathcal{N}$, but their categorical structures are similar whenever two η's give the same Θ. If $\Theta = \Pi$, we say that η is *non-degenerate*, in which case $\mathcal{N}_{\theta,\eta}$ is semisimple with one irreducible object, and $\mathcal{N}_\eta$ is equivalent to the category of finite dimensional $\mathcal{Z}(\mathfrak{g})$-modules [MS14, Theorem 5.6 and 5.9]. If $\Theta = \varnothing$ (i.e. $\eta = 0$), $\mathcal{N}_{\theta,0}$ recovers the category of *highest weight modules*, which is also known as Bernstein-Gelfand-Gelfand's category $\mathcal{O}$ (although their original definition of category $\mathcal{O}$ is slightly different).

We aim to describe characters of irreducible objects in $\mathcal{N}_{\theta,\eta}$ for any θ and η in terms of characters of certain standard modules $M(\lambda, \eta)$ which are constructed by McDowell analogous to Verma modules. We first describe its definition in the non-degenerate case. The *cyclic Whittaker module* for λ and non-degenerate η is

$$Y_\mathfrak{g}(\lambda, \eta) = \mathcal{U}(\mathfrak{g})_\lambda \underset{\mathcal{U}(\mathfrak{n})}{\otimes} \mathbb{C}_\eta,$$

where $\mathfrak{n}$ acts on $\mathbb{C}_\eta$ by η. Kostant [Kos78, Theorem A] showed that $Y_\mathfrak{g}(\lambda,\eta)$ is irreducible and Miličić-Soergel [MS14, Theorem 5.6] showed that it is the unique irreducible object of the semisimple category $\mathcal{N}_{\theta,\eta}$. When η is degenerate, the same definition produces a possibly decomposable module. Instead, we take the standard module to be the one parabolically induced from the cyclic Whittaker module of a Levi subalgebra. More precisely, let $\mathfrak{p}_\Theta$ be the parabolic subalgebra of type Θ containing $\mathfrak{b}$ with ad $\mathfrak{h}$-stable Levi $\mathfrak{l}_\Theta$. η then restricts to a non-degenerate character of $\mathfrak{l}_\Theta \cap \mathfrak{n}$. The **standard Whittaker module** is

$$M(\lambda,\eta) = \mathcal{U}(\mathfrak{g}) \underset{\mathcal{U}(\mathfrak{p}_\Theta)}{\otimes} Y_{\mathfrak{l}_\Theta}(\lambda - \rho + \rho_\Theta, \eta).$$

When η is non-degenerate, $M(\lambda,\eta) = Y_\mathfrak{g}(\lambda,\eta)$; when $\eta = 0$, these are just Verma modules.

$M(\lambda,\eta)$ lands in the category $\mathcal{N}_{\theta,\eta}$. Also, $M(\lambda_1,\eta) \cong M(\lambda_2,\eta)$ if and only if λ_1 and λ_2 are in the same W_Θ-orbit. Therefore, if $\lambda \in \theta$ is fixed and regular (and will be chosen to be antidominant with respect to roots in $\mathfrak{b}$ in this section and the next), standard modules in $\mathcal{N}_{\theta,\eta}$ are parameterized by right W_Θ-cosets in W. If λ is singular, then standard objects are instead parameterized by double cosets $W_\Theta \backslash W / W^\lambda$, where W^λ is the stabilizer of λ. For a right W_Θ-coset C, we will write w^C for the unique longest element in C under the Bruhat order and write $M(w^C\lambda,\eta)$ for the corresponding standard module. McDowell showed that each $M(w^C\lambda,\eta)$ has a unique irreducible quotient, denoted by

$$L(w^C\lambda,\eta).$$

Any irreducible object in $\mathcal{N}_{\theta,\eta}$ arises in this way, and $L(w^C\lambda,\eta) = L(w^D\lambda,\eta)$ if and only if $M(w^C\lambda,\eta) = M(w^D\lambda,\eta)$. So irreducible objects are also parameterized by $W_\Theta\backslash W$ if λ is regular and by $W_\Theta \backslash W / W^\lambda$ if λ is singular. The irreducible objects and standard objects form two natural bases of Grothendieck group $K\mathcal{N}_{\theta,\eta}$.

By mimicking the construction for Verma modules, Romanov developed in her dissertation [Rom21, §2.2] a character theory for $\mathcal{N}_{\theta,\eta}$. This is a map ch on objects of $\mathcal{N}_{\theta,\eta}$ that factors through and is injective on the Grothendieck group $K\mathcal{N}_{\theta,\eta}$. The characters of standard Whittaker modules are computed explicitly in *loc. cit.* (see [Rom21, §2.2 Equation (2)]). Although our main results are stated in terms of the character map, they are in fact statements of the Grothendieck group, and we will not use any other property of the character map.

Nevertheless, let us briefly describe the shape of this character theory. Let $\mathfrak{h}^\Theta$be the center of $\mathfrak{l}_\Theta$, let $\mathfrak{s}_\Theta = [\mathfrak{l}_\Theta, \mathfrak{l}_\Theta]$be the semisimple part of $\mathfrak{l}_\Theta$, and let $\mathfrak{h}_\Theta = \mathfrak{s}_\Theta \cap \mathfrak{h}$,be a Cartan in $\mathfrak{s}_\Theta$, so that $\mathfrak{h} = \mathfrak{h}_\Theta \oplus \mathfrak{h}^\Theta$. Since η is non-degenerate on $\mathfrak{s}_\Theta \cap \mathfrak{n}$, the category $\mathcal{N}(\mathfrak{s}_\Theta)_\eta$ of Whittaker modules of $\mathfrak{s}_\Theta$ with generalized $\mathfrak{s}_\Theta \cap \mathfrak{n}$-character η is equivalent to the category of finite dimensional $\mathcal{Z}(\mathfrak{s}_\Theta)$-modules.

The Grothendieck group $K\mathcal{N}(\mathfrak{s}_\Theta)_\eta$ is therefore free abelian with a basis given by dominant integral weights of $\mathfrak{h}_\Theta$. For an object $V \in \mathcal{N}(\mathfrak{s}_\Theta)_\eta$, we write $[V]$ for its class in $K\mathcal{N}(\mathfrak{s}_\Theta)_\eta$.

Any object V in $\mathcal{N}_\eta$ is necessarily locally $\mathfrak{h}^\Theta$-finite. Hence V can be decomposed into a direct sum of generalized $\mathfrak{h}^\Theta$-weight spaces $V^{\underline{\mu}}$, $\underline{\mu} \in (\mathfrak{h}^\Theta)^*$. It can be shown that each one of these is an $\mathfrak{s}_\Theta$-module living in $\mathcal{N}(\mathfrak{s}_\Theta)_\eta$. The character map is defined by

$$\mathrm{ch} : \mathrm{Obj}\,\mathcal{N}_{\theta,\eta} \longrightarrow K\mathcal{N}(\mathfrak{s}_\Theta)_\eta \underset{\mathbb{Z}}{\otimes} \mathbb{Z}[[(\mathfrak{h}^\Theta)^*]], \quad V \mapsto \sum_{\underline{\mu} \in (\mathfrak{h}^\Theta)^*} [V^{\underline{\mu}}] e^{\underline{\mu}},$$

where $\mathbb{Z}[[(\mathfrak{h}^\Theta)^*]]$ is the group of power series in $e^{\underline{\mu}}$, $\underline{\mu} \in (\mathfrak{h}^\Theta)^*$. The characters of standard modules are easily computed, and is a linear combination with partition functions as coefficients, similar to Verma modules. The readers can refer to [Rom21, §2.2 Equation (2)] for details.

1.2 The character formula

Throughout this dissertation, we will use a subscript λ on the combinatorial objects defined in the previous section to denote subobjects that are integral to λ. Thus Σ_λ consists of roots $\alpha \in \Sigma$ integral to λ, meaning $\alpha^\vee(\lambda) \in \mathbb{Z}$, where $\alpha^\vee$ is the coroot of α; $\Sigma_\lambda^+ = \Sigma_\lambda \cap \Sigma^+$, and $\Pi_\lambda \subseteq \Sigma_\lambda^+$ is the corresponding set of simple roots (which may not be simple in Σ^+); W_λ is the Weyl group of Σ_λ, which can be embedded in W as $\{w \in W \mid w\lambda - \lambda \in \mathbb{Z} \cdot \Sigma\}$.

Let us fix a λ that is antidominant regular with respect to Σ^+. This means $\alpha^\vee(\lambda)$ is not zero or a positive integer for all $\alpha \in \Sigma^+$. We aim to express the character of $L(w^C\lambda, \eta)$ in terms of the characters of standard modules $M(w^D\lambda, \eta)$. The precise expression involves combinatorial data extracted from double cosets $W_\Theta \backslash W / W_\lambda$. Each double coset $W_\Theta u W_\lambda$ contains a unique shortest element u with respect to Bruhat order (Corollary 2.3.3). We can then take the intersections of uW_λ with various right W_Θ-cosets in $W_\Theta u W_\lambda$. This produces a partition of uW_λ. Left-translating back into W_λ, we obtain a partition of W_λ, which coincides with the partition given by right $W_{\lambda,\Theta(u,\lambda)}$-cosets of W_λ (Proposition 2.4.3). Here, $W_{\lambda,\Theta(u,\lambda)}$ is a parabolic subgroup of W_λ corresponding to the subset of simple roots $\Theta(u,\lambda) = u^{-1}\Sigma_\Theta \cap \Pi_\lambda \subseteq \Pi_\lambda$ (Proposition 2.4.2). We thus obtain a map from the set of right W_Θ-cosets in $W_\Theta u W_\lambda$ to the set of right $W_{\lambda,\Theta(u,\lambda)}$-cosets in W_λ, i.e. a map

$$(-)|_\lambda : W_\Theta \backslash W_\Theta u W_\lambda \to W_{\lambda,\Theta(u,\lambda)} \backslash W_\lambda \tag{1.2.1}$$

(Notation 2.4.5). Recall that there is a partial order $\leqslant$ on $W_\Theta \backslash W$ inherited from the restriction of Bruhat order to the set of the longest element in each coset (see §2.2). We denote the partial order on $W_{\lambda,\Theta(u,\lambda)} \backslash W_\lambda$ by $\leqslant_{u,\lambda}$.

The double cosets reflects the block decomposition of $\mathcal{N}_{\theta,\eta}$ (here a "block" means an indecomposable direct summand of $\mathcal{N}_{\theta,\eta}$). On the level of character formula, $\operatorname{ch} M(w^D\lambda, \eta)$ appears in $\operatorname{ch} L(w^C\lambda, \eta)$ only if D and C are in the same double coset $W_\Theta u W_\lambda$ and $D|_\lambda \leqslant_{u,\lambda} C|_\lambda$ (for which we will simply write $D \leqslant_{u,\lambda} C$; see Notation 2.4.5). The precise coefficient of $\operatorname{ch} M(w^D\lambda, \eta)$ is described by Whittaker Kazhdan-Lusztig polynomials. For a triple (W, Π, Θ), **Whittaker Kazhdan-Lusztig polynomials** are polynomials $P_{CD} \in \mathbb{Z}[q]$ labeled by pairs (C, D) of right W_Θ-cosets with $C \leqslant D$ (defined in 4.1.1). By Romanov's work, these polynomials compute (at $q = -1$) the character formula of irreducible Whittaker modules for integral infinitesimal characters. Applied to the triple $(W_\lambda, \Pi_\lambda, \Theta(u, \lambda))$ (see 4.1.2, or (W.1) and (W.2) in §1.4) and the pair (C, D), we obtain polynomials $P^{u,\lambda}_{CD} = P^{u,\lambda}_{C|_\lambda, D|_\lambda}$.

Theorem 1.2.2 (Character formula: regular case)**.** *Let λ be antidominant regular. For any $C \in W_\Theta\backslash W$, let $W_\Theta u W_\lambda$ be the double coset containing C, where u is the unique shortest element in this double coset. Then*

$$\operatorname{ch} L(w^C\lambda, \eta) = \operatorname{ch} M(w^C\lambda, \eta) + \sum_{\substack{D \in W_\Theta\backslash W_\Theta u W_\lambda \\ D <_{u,\lambda} C}} P^{u,\lambda}_{CD}(-1) \operatorname{ch} M(w^D\lambda, \eta),$$

This appears as Theorem 5.1.2 below. We also extend this to singular λ in Theorem 5.2.5. At the special case $\eta = 0$, we recover the non-integral Kazhdan-Lusztig conjecture for Verma modules.

The above formula follows from an algorithm (so called Kazhdan-Lusztig algorithm), namely Theorem 4.2.2. The proof of the algorithm is done by studying (weakly) equivariant $\mathcal{D}$-modules. Moreover, our tool for dealing with non-integrality (namely the non-integral intertwining functor) is also geometric and makes sense for arbitrary (possibly non-equivariant) quasi-coherent $\mathcal{D}$-modules. Therefore the method used here for extending integral results to the non-integral case should apply to other Kazhdan-Lusztig problems as well. We postpone the statement of the algorithm to §1.4 after discussing the geometric ideas behind the proof.

1.3 Localization of Whittaker modules

The strategy of Miličić-Soergel, Romanov, and the author is to study $\mathcal{D}$-modules corresponding to Whittaker modules. In this section we introduce the localization framework related to Whittaker modules. References for facts below include [BB81], [BB93], [MS14], [Mil], [Rom21].

Let X be the flag variety of $\mathfrak{g}$, the variety of Borel subalgebras of $\mathfrak{g}$. The sheaf of ordinary (algebraic) differential operators $\mathcal{D}_X$ is the subsheaf of $\mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)$ generated by multiplications of functions and actions of vector fields. The natural action of G on X can be differentiated, which assigns each element in $\mathfrak{g}$ a vector field on X, whence a map $\mathfrak{g} \to \mathcal{D}_X$.

More generally, for each $\lambda \in \mathfrak{h}^*$, Beilinson-Bernstein constructed in [BB81] a twisted sheaf of differential operators $\mathcal{D}_\lambda$ on X together with a map $\mathfrak{g} \to \mathcal{D}_\lambda$ that induces an isomorphism $\mathcal{U}_\theta \cong \Gamma(X, \mathcal{D}_\lambda)$ ($\mathcal{U}_\theta$ is defined in §1.1). Here $\mathcal{D}_\lambda$ is a sheaf of $\mathbb{C}$-algebras that is locally isomorphic to $\mathcal{D}_X$. We use the parametrization of these sheaves as in [Mil, Chapter 2 §1], under which $\mathcal{D}_X = \mathcal{D}_{-\rho}$. $\lambda \in \mathfrak{h}^*$ is said to be *antidominant* if for all $\alpha \in \Sigma^+$, the coroot $\alpha^\vee$ satisfies $\alpha^\vee(\lambda) \notin \mathbb{Z}_{>0}$; *regular* if $\alpha^\vee(\lambda) \neq 0$ for all α. If λ is antidominant and regular, Beilinson and Bernstein showed that taking global sections on X is an equivalence of categories

$$\Gamma(X, -) : \mathrm{Mod}_{qc}(\mathcal{D}_\lambda) \cong \mathrm{Mod}(\mathcal{U}_\theta) \tag{1.3.1}$$

between the category of quasi-coherent $\mathcal{D}_\lambda$-modules and the category of $\mathcal{U}_\theta$-modules, and a quasi-inverse is given by the *localization* functor $\mathcal{D}_\lambda \otimes_{\mathcal{U}_\theta} -$. If λ is only antidominant but not regular, $\Gamma(X, -)$ is still exact, but some $\mathcal{D}_\lambda$-modules can have zero global section. The subcategory $\mathcal{N}_{\theta,\eta}$ of $\mathrm{Mod}(\mathcal{U}_\theta)$ corresponds, under the above equivalence of categories, to the subcategory $\mathrm{Mod}_{coh}(\mathcal{D}_\lambda, N, \eta)$ consisting of η**-twisted Harish-Chandra sheaves**. This is the full subcategory of all coherent $\mathcal{D}_\lambda$-modules consisting of those $\mathcal{V}$ such that

- $\mathcal{V}$ is an N-equivariant $\mathcal{O}_X$-module,
- the action map $\mathcal{D}_\lambda \otimes \mathcal{V} \to \mathcal{V}$ of $\mathcal{D}_\lambda$ on $\mathcal{V}$ is N-equivariant, and
- for all $n \in \mathfrak{n}$, the equation $\pi(\xi) = \mu(\xi) + \eta(\xi)$ holds in $\mathrm{End}_{\mathbb{C}}(\mathcal{V})$, where π is the action of $\mathfrak{n}$ induced by $\mathfrak{n} \subset \mathfrak{g} \to \mathcal{D}_\lambda \circlearrowright \mathcal{V}$, and μ is the action given by the differential of the N-equivariant structure on $\mathcal{V}$.

η-twisted Harish-Chandra sheaves are automatically *holonomic* (see [MS14, Lemma 1.1] for a proof; see [HTT08, 2.3.6] for the definition of holonomicity and [HTT08, Chapter 3] for properties of holonomic modules). Holonomic modules share very nice properties. They have finite length (which, in particular, implies the finite length result of McDowell). They are preserved by direct images and inverse images along morphisms of smooth varieties. They admit a duality operation

$$\mathbb{D} : \mathrm{Mod}_{hol}(\mathcal{D}_\lambda) \xrightarrow{\sim} \mathrm{Mod}_{hol}(\mathcal{D}_\lambda^{op}) \cong \mathrm{Mod}_{hol}(\mathcal{D}_{-\lambda})$$

(here Mod_{hol} denotes the category of holonomic modules, and the last isomorphism is because ${\mathcal{D}_\lambda}^{op} \cong \mathcal{D}_{-\lambda}$). In our notations for holonomic $\mathcal{D}$-modules, for a morphism f between smooth varieties, we have direct images f_+, $f_!$ and inverse images f^+, $f^!$. Here f_+ agrees with the one in [BGK$^+$87, VI.5] and with $\int_f$ in [HTT08]. It also agrees with the $*$-direct image in the usual six-functor formalism. $f_!$ is the functor obtained by conjugating f_+ by holonomic duality $\mathbb{D}$ (this is denoted by $\int_{f!}$ in [HTT08]). $f^!$ agrees with the one defined in [BGK+87, VI.4] ($f^\dagger$ in [HTT08]). When f is a closed immersion of a smooth subvariety, $H^0 f^! \mathcal{V}$ consists of sections of $\mathcal{V}$ supported in the subvariety. f^+ is a shift of $f^!$ by the relative dimension (f^* in [HTT08]); forgetting the $\mathcal{D}$-module structures, f^+ agrees with the usual $\mathcal{O}$-module inverse image f^*. η-twisted Harish-Chandra sheaves are functorial with respect to all these operations.

Let $C(w)$, $w \in W$ be the Schubert cells (i.e. N-orbits) on X, with inclusion maps $i_w : C(w) \to X$. There exist nonzero η-twisted Harish-Chandra sheaves on $C(w)$ if and only if $w = w^C$ is the longest element in the right W_Θ-coset that contains it. If this is the case, the category $\mathrm{Mod}_{coh}(\mathcal{D}_{C(w^C)}, N, \eta)$ is semisimple, in which the unique irreducible object, denoted by $\mathcal{O}^\eta_{C(w^C)}$, has $\mathcal{O}_{C(w^C)}$ as the underlying structure of an N-equivariant $\mathcal{O}_{C(w^C)}$-module, but with an η-twisted $\mathcal{D}_{C(w^C)}$-action (Lemma 3.2.1 or [MS14, §4]). We call the $\mathcal{D}$-module direct images

$$\mathcal{I}(w^C, \lambda, \eta) = i_{w^C +} \mathcal{O}^\eta_{C(w^C)}, \qquad \mathcal{M}(w^C, \lambda, \eta) = i_{w^C !} \mathcal{O}^\eta_{C(w^C)}$$

the **standard module** and the **costandard module**, respectively. The standard module $\mathcal{I}(w^C, \lambda, \eta)$ contains a unique irreducible submodule

$$\mathcal{L}(w^C, \lambda, \eta),$$

and $\mathcal{L}(w^C, \lambda, \eta)$ is the unique irreducible quotient of $\mathcal{M}(w^C, \lambda, \eta)$. The $\mathcal{L}(w^C, \lambda, \eta)$'s exhaust all irreducible objects in $\mathrm{Mod}_{coh}(\mathcal{D}_\lambda, N, \eta)$ ([MS14, §3]). Romanov showed (using the character theory she developed) that if λ is antidominant, $\Gamma(X, -)$ sends $\mathcal{M}(w^C, \lambda, \eta)$ to $M(w^C\lambda, \eta)$ and $\mathcal{L}(w^C, \lambda, \eta)$ to either $L(w^C\lambda, \eta)$ or 0. If λ is furthermore regular, $\mathcal{L}(w^C, \lambda, \eta)$ is always sent to $L(w^C\lambda, \eta)$. This allows us to study Whittaker modules using geometry on X.

In practice $\mathcal{I}(w^C, \lambda, \eta)$ is more convenient to work with than $\mathcal{M}(w^C, \lambda, \eta)$. The holonomic duality $\mathbb{D}$ sends $\mathcal{I}(w^C, \lambda, \eta)$ and $\mathcal{L}(w^C, \lambda, \eta)$ to $\mathcal{M}(w^C, -\lambda, \eta)$ and $\mathcal{L}(w^C, -\lambda, \eta)$, respectively. So we have the following flowchart

$$\mathcal{N}_{\theta,\eta} \xrightarrow{\mathcal{D}_\lambda \otimes \mathcal{U}_\theta -} \mathrm{Mod}_{coh}(\mathcal{D}_\lambda, N, \eta) \xrightarrow{\mathbb{D}} \mathrm{Mod}_{coh}(\mathcal{D}_{-\lambda}, N, \eta),$$

$$L(w^C\lambda, \eta) \mapsto \mathcal{L}(w^C, \lambda, \eta) \mapsto \mathcal{L}(w^C, -\lambda, \eta),$$

$$M(w^C\lambda, \eta) \mapsto \mathcal{M}(w^C, \lambda, \eta) \mapsto \mathcal{I}(w^C, -\lambda, \eta).$$

Because of the finite length property, the set of irreducible objects form a basis for the Grothendieck group $K\,\mathrm{Mod}_{coh}(\mathcal{D}_{-\lambda}, N, \eta)$. A standard argument using pullback-pushforward adjunctions shows that the set of standard modules also form a basis for $K\,\mathrm{Mod}_{coh}(\mathcal{D}_{-\lambda}, N, \eta)$. Therefore, our goal of finding coefficients of $\mathrm{ch}\, M(w^D\lambda, \eta)$ in $\mathrm{ch}\, L(w^C\lambda, \eta)$ is the same as finding the change of bases matrix in $K\,\mathrm{Mod}_{coh}(\mathcal{D}_{-\lambda}, N, \eta)$ from the basis given by the $\mathcal{L}$'s to the one given by the $\mathcal{I}$'s. Of course, the special case of $\eta = 0$ has already been treated by the ordinary Kazhdan-Lusztig conjecture, proven by Beilinson-Bernstein [BB81] and Brylinsky-Kashiwara [BK81].

1.4 The Kazhdan-Lusztig algorithms

Before discussing the extension to non-integral infinitesimal characters, let us first discuss Romanov's work in the integral case. Her argument is in the same spirit as the algorithm for highest weight modules which we now recall.

Consider the case $\eta = 0$ of highest weight modules. The conjecture of Kazhdan-Lusztig [KL79] predicts that the change of basis between Verma modules and irreducibles are computed by combinatorics in the Hecke algebra $\mathcal{H}$. To relate our problem with $\mathcal{H}$, we would like to build a comparision map ν that fits into the commutative diagram

$$\begin{array}{ccc} \mathrm{Mod}_{coh}(\mathcal{D}_\lambda, N) & \xrightarrow{\nu} & \mathcal{H} \\ {\scriptstyle [-]}\downarrow & & \downarrow{\scriptstyle q=-1} \\ K\,\mathrm{Mod}_{coh}(\mathcal{D}_\lambda, N) & \xrightarrow{\cong} & \mathbb{Z}[W] \end{array}.$$

In this diagram, $\mathcal{H}$ and $\mathbb{Z}[W]$ are the Hecke algebra and the group algebra of W, respectively, and the bottom map sends $[\mathcal{I}_w]$ to the basis in $\mathbb{Z}[W]$ labeled by w. Moreover, the regular action of $\mathcal{H}$ on the top right corner should lift to an $\mathcal{H}$ "action" on the $\mathcal{I}_w$ and $\mathcal{L}_w$ in $\mathrm{Mod}_{coh}(\mathcal{D}_\lambda, N)$. Once this diagram is constructed, $[\mathcal{L}_w] = \nu(\mathcal{L}_w)|_{q=-1}$ by commutativity of the diagram, and $\nu(\mathcal{L}_w)$ can be computed by studying the $\mathcal{H}$-action.

In further detail, recall that the Hecke algebra $\mathcal{H}$ is an algebra which has an underlying free $\mathbb{Z}[q^{\pm 1}]$-module structure with two bases labeled by W: the defining basis $\{\delta_w\}$ and the Kazhdan-Lusztig basis $\{C_w\}$ [KL79]. The Kazhdan-Lusztig basis is characterized by three conditions:

(KL.1) the expansion of C_w in terms of the δ_v's involve only those with $v \leqslant w$, the coefficient of δ_w is 1, and the coefficient of δ_v ($v < w$) is a polynomial $P_{wv}(q)$ with no constant term;

(KL.2) the product $C_w C_s$, where s is a simple reflection so that $ws > w$, is a $\mathbb{Z}$-linear combination of C_v's with $v \leqslant ws$;

(KL.3) $C_s = \delta_s + q$

(after some normalizations, the first two conditions are (1.1.b) and (2.3.b) of [KL79], respectively). Here $<$ and $\leqslant$ are the Bruhat order on W. These conditions inductively determine the Kazhdan-Lusztig basis and provide a recursive algorithm for computing it. The coefficients P_{wv} of the δ_v's are the famous *Kazhdan-Lusztig polynomials*. The Kazhdan-Lusztig conjecture predicts that the coefficients of the Verma modules in the irreducible modules in the Grothendieck group are given by Kazhdan-Lusztig polynomials evaluated at -1 (or at 1, depending on the normalization). In view of the above diagram, proving the conjecture amounts to constructing ν so that $\nu(\mathcal{I}_w) = \delta_w$ and $\nu(\mathcal{L}_w) = C_w$.

To this end, we define the map ν by sending a $\mathcal{D}_\lambda$-module $\mathcal{F}$ to a linear combination of δ_v's where the coefficient of δ_v is the generating function (in variable q) of the pullback of $\mathcal{F}$ to the Schubert cell $C(v)$:

$$\nu(\mathcal{F}) = \sum_{w \in W} \left(\chi_q i_w^! \mathcal{F}\right) \delta_w.$$

Then ν sends $\mathcal{I}_v$ to δ_v, and $\nu(\mathcal{L}_w)$ automatically satisfies condition (KL.1) for support reason. Moreover, multiplication by C_s on δ_w for a simple reflection s lifts on $\mathcal{I}_w$ to the "push-pull" operation along the natural map $X \to X_s$ to the type-s partial flag variety (we call this operation the U-functor since it agrees with the one defined in [Vog79, Definition 3.8]). (KL.2) is proven by an induction on $\ell(w)$ by showing the same lifting for irreducibles, using the Decomposition Theorem of Beilinson-Bernstein-Deligne [BBD82] for regular holonomic $\mathcal{D}$-modules (or perverse sheaves). This proves $\nu(\mathcal{L}_w) = C_w$ and hence the Kazhdan-Lusztig conjecture. A detailed argument following these lines can be found in Miličić's unpublished notes [Mil, Chapter 5]. Since the character map on highest weight modules factors through the Grothendieck group, one can write down characters of irreducible modules in terms of characters of Verma modules, and the latter can be easily computed.

This proof naturally extends to parabolic highest weight categories corresponding to a subset Θ of simple roots and with regular integral infinitesimal characters. Two bases of the Grothendieck group are now given by parabolic Verma modules and their irreducible quotients, both labeled by right W_Θ-cosets. The map ν is now defined by pulling back to orbits of a parabolic subgroup P_Θ of type Θ, and the image of the comparison map ν is now replaced by a smaller $\mathcal{H}$-module. The Kazhdan-Lusztig polynomials are then replaced by *parabolic Kazhdan-Lusztig polynomials*, which form a subset of the ordinary Kazhdan-Lusztig polynomials.

In the case of Whittaker modules with integral regular infinitesimal characters, we still have two bases of the Grothendieck group labeled by right W_Θ-cosets: the standard Whittaker modules defined by McDowell and their irreducible quotients. By the work of Miličić-Soergel [MS97], the category $\mathcal{N}_{\theta,\eta}$ is equivalent to the highest weight category with a singular infinitesimal character. The latter is known to be Koszul dual to parabolic highest weight category with an integral regular infinitesimal character by the work of Beilinson-Ginzburg-Soergel [BGS96]. Therefore, the Kazhdan-Lusztig polynomials of Whittaker modules (what Romanov called *Whittaker Kazhdan-Lusztig polynomials*) are expected to be dual to parabolic Kazhdan-Lusztig polynomials. More precisely, if we define Θ as in (1.1.1), then the Whittaker category $\mathcal{N}_{\theta,\eta}$ are expected to be dual to the parabolic highest weight category determined by Θ. A starting point towards proving this would be a Kazhdan-Lusztig algorithm for Whittaker modules. However, the $\mathcal{D}$-modules in this situation are no longer regular holonomic (merely holonomic). Therefore a decomposition theorem for general holonomic modules is needed in order for the same argument to work. This is proven by Mochizuki [Moc11]. Romanov then adapted the strategy for highest weight modules to the case of Whittaker modules in her dissertation (later published in [Rom21]) and obtained a Kazhdan-Lusztig algorithm. Together with the character theory she developed, this implies a character formula for irreducible Whittaker modules. The comparison map ν in the highest weight setting now becomes a map

$$\mathrm{Mod}_{coh}(\mathcal{D}_\lambda, N, \eta) \xrightarrow{\nu} \mathcal{H}_\Theta$$

defined by pulling back a sheaf to orbits of the form $C(w^C)$, where $\mathcal{H}_\Theta$ is an $\mathcal{H}$-module which is free over $\mathbb{Z}[q^{\pm 1}]$ with a basis labeled by $W_\Theta \backslash W$. This $\mathcal{H}$-module structure defines a Kazhdan-Lusztig basis of $\mathcal{H}_\Theta$, whose elements coincide with the images of irreducible $\mathcal{D}$-modules under ν.

The work of this paper generalizes Romanov's algorithm to arbitrary infinitesimal characters. There are two extra complications compared to Romanov's situation. First, although standard and irreducible Whittaker modules are still parameterized by $W_\Theta \backslash W$, now our category is a direct sum of smaller blocks, and different blocks have different sizes. On the other hand, the parabolic highest weight category can have fewer blocks, so the duality mentioned in the preceding paragraph fails. Nevertheless, one can expect the blocks to be parameterized by Weyl group data involving both W_Θ and W_λ. Indeed, as can be seen from the character formula 1.2.2, blocks are parameterized by double cosets $W_\Theta \backslash W / W_\lambda$, and the polynomials for each block turn out to be the same as (integral) Whittaker Kazhdan-Lusztig polynomials of the integral Weyl group W_λ.

The second complication is that the "push-pull" operation along $X \to X_s$ does not exist when λ is non-integral to s – there is no sheaf of twisted differential operators on X_s that pulls back to $\mathcal{D}_\lambda$. As a result, induction on $\ell(w)$ cannot proceed as before. To remedy this, we use the *intertwining functor* I_s for non-integral s in place of the U-functor. It is an equivalence of categories between $\mathcal{D}_\lambda$-modules and $\mathcal{D}_{s\lambda}$-modules. This allows us to increase $\ell(w)$ and retain induction hypotheses. This idea of proof is suggested to the author by Miličić.

We can now state our algorithm. We fix a character $\eta : \mathfrak{n} \to \mathbb{C}$ and define a subset Θ of simple from η as in (1.1.1). For each λ (not necessarily antidominant), we define a map ν similar to the highest weight case, but now we only pull back to Schubert cells of the form $C(w^C)$. It fits into the commutative diagram

$$\begin{array}{ccccc} \mathrm{Mod}_{coh}(\mathcal{D}_\lambda, N, \eta) & \xrightarrow{\nu} & \mathcal{H}_\Theta & \xrightarrow{(-)|_\lambda} & \bigoplus_{W_\Theta u W_\lambda} \mathcal{H}_{\Theta(u,\lambda)} \\ \downarrow{[-]} & & \downarrow{q=-1} & & \downarrow{q=-1} \\ K\,\mathrm{Mod}_{coh}(\mathcal{D}_\lambda, N, \eta) & \xrightarrow{\cong} & \mathbb{Z}[W_\Theta \backslash W] & \xrightarrow{(-)|_\lambda} & \bigoplus_{W_\Theta u W_\lambda} \mathbb{Z}[W_{\lambda,\Theta(u,\lambda)} \backslash W_\lambda] \end{array}.$$

Here $\mathcal{H}_\Theta$ is the free $\mathbb{Z}[q^{\pm 1}]$-module with basis $\{\delta_C\}_{C \in W_\Theta \backslash W}$, $\mathbb{Z}[W_\Theta \backslash W]$ is the $\mathbb{Z}$-module with the same basis, and the first horizontal map at the bottom sends $[\mathcal{I}(w^C, \lambda, \eta)]$ to δ_C. The modules $\mathcal{H}_{\Theta(u,\lambda)}$ and $\mathbb{Z}[W_{\lambda,\Theta(u,\lambda)} \backslash W_\lambda]$ are defined similarly but their bases are instead labeled by $W_{\lambda,\Theta(u,\lambda)} \backslash W_\lambda$. The map $(-)|_\lambda$ is defined on basis elements analogous to (1.2.1). Each $\mathcal{H}_{\Theta(u,\lambda)}$ is a module over the Hecke algebra $\mathcal{H}_\lambda = \mathcal{H}(W_\lambda)$ of the integral Weyl group W_λ. Thus each $\alpha \in \Pi_\lambda$ defines an operator $T_\alpha^{u,\lambda}$ on $\mathcal{H}_{\Theta(u,\lambda)}$ representing the multiplication of the Kazhdan-Lusztig basis element $C_{\lambda,s_\alpha} \in \mathcal{H}_\lambda$ corresponding to the simple reflection s_α. Romanov's main result [Rom21, Theorem 11], interpreted combinatorially and applied to $\mathcal{H}_{\Theta(u,\lambda)}$, says that the operators $T_\alpha^{u,\lambda}$ inductively define a Kazhdan-Lusztig basis of $\mathcal{H}_{\Theta(u,\lambda)}$ in a similar fashion as the condition (KL.2). More precisely, the **Kazhdan-Lusztig basis** $\{\psi_{u,\lambda}(F)\}$ of $\mathcal{H}_{\Theta(u,\lambda)}$ is the unique basis such that

(W.1) $\psi_{u,\lambda}(F) = \delta_F + \sum_{G <_{u,\lambda} F} P_{FG}^{u,\lambda}\, \delta_G$ for some $P_{FG}^{u,\lambda} \in q\mathbb{Z}[q]$; and

(W.2) if F is not the shortest right coset, there exist $\alpha \in \Pi_\lambda$ and $c_G \in \mathbb{Z}$ such that $Fs_\alpha <_{u,\lambda} F$ and

$$T_\alpha^{u,\lambda}(\psi_{u,\lambda}(Fs_\alpha)) = \sum_{G \leqslant_{u,\lambda} F} c_G\, \psi_{u,\lambda}(G)$$

(see 4.1.2). We can still formally consider an $\mathcal{H}$-module structure on $\mathcal{H}_\Theta$ as in the integral case and define operators $T_\alpha : \mathcal{H}_\Theta \to \mathcal{H}_\Theta$for simple roots α. When α is integral, T_α represents the U-functor, preserves the decomposition $(-)|_\lambda$, and restricts to $T_\alpha^{u,\lambda}$ on each $\mathcal{H}_{\Theta(u,\lambda)}$. When a simple root β is non-integral, we will instead consider the endomorphism $(-) \cdot s_\beta$ on $\mathcal{H}_\Theta$ given by $\delta_C \cdot s_\beta = \delta_{Cs_\beta}$, which represents the intertwining functor I_{s_β}.

Here is our (slightly rephrased) algorithm.

Theorem 1.4.1 (Kazhdan-Lusztig algorithm)**.** *Fix a character $\eta : \mathfrak{n} \to \mathbb{C}$. For any λ and any $C \in W_\Theta \backslash W$, write $W_\Theta u W_\lambda$ for the double coset containing C, where u is the unique shortest element in this double coset. Then*

(A.1) There exist polynomials $P^{u,\lambda}_{CD} \in q\mathbb{Z}[q]$ *so that*

$$\nu(\mathcal{L}(w^C,\lambda,\eta)) = \nu(\mathcal{I}(w^C,\lambda,\eta)) + \sum_{\substack{D\in W_\Theta\backslash W_\Theta u W_\lambda \\ D<_{u,\lambda} C}} P^{u,\lambda}_{CD}\ \nu(\mathcal{I}(w^D,\lambda,\eta)).$$

(A.2) For any integral simple root α *such that* $Cs_\alpha < C$*, there exist integers* c_D *depending on* C*,* D*, and* s_α*, such that*

$$T_\alpha(\nu(\mathcal{L}(w^{Cs_\alpha},\lambda,\eta))) = \sum_{\substack{D\in W_\Theta\backslash W_\Theta u W_\lambda \\ D\leqslant_{u,\lambda} C}} c_D\ \nu(\mathcal{L}(w^D,\lambda,\eta)).$$

(A.3) For any non-integral simple root β *such that* $Cs_\beta < C$*,*

$$\nu(\mathcal{L}(w^C,\lambda,\eta))\cdot s_\beta = \nu(\mathcal{L}(w^{Cs_\beta},s_\beta\lambda,\eta)).$$

(A.4) $\nu(\mathcal{L}(w^C,\lambda,\eta))|_\lambda$ *is a Kazhdan-Lusztig basis element of* $\mathcal{H}_{\Theta(u,\lambda)}$*.*

This appears as Theorem 4.2.2 below. The character formula 1.2.2 follows by taking (A.1) and (A.4) for $-\lambda$ dominant regular (so that λ is antidominant regular), precomposing ν with $\mathbb{D}$ (so that the $\mathcal{I}$'s become the $\mathcal{M}$'s), descending to the Grothendieck group by specializing at $q=-1$, passing through Beilinson-Bernstein localization, and applying the character map.

The proof of the algorithm is an induction on the length $\ell(w^C)$. The ideas behind (A.1) and (A.2) are similar to (KL.1) and (KL.2), respectively. (A.3) reflects the action of non-integral intertwining functor I_{s_β}. In fact, the following diagram commutes

$$\begin{array}{ccccc}
\mathrm{Mod}_{coh}(\mathcal{D}_\lambda,N,\eta) & \xrightarrow{\nu} & \mathcal{H}_\Theta & \xrightarrow{(-)|_\lambda} & \bigoplus \mathcal{H}_{\Theta(u,\lambda)} \\
\downarrow{\scriptstyle I_{s_\beta}} & & \downarrow{\scriptstyle (-)\cdot s_\beta} & & \downarrow{\scriptstyle s_\beta\cdot(-)\cdot s_\beta} \\
\mathrm{Mod}_{coh}(\mathcal{D}_{s_\beta\lambda},N,\eta) & \xrightarrow{\nu} & \mathcal{H}_\Theta & \xrightarrow{(-)|_\lambda} & \bigoplus \mathcal{H}_{\Theta(r,s_\beta\lambda)}
\end{array}$$

(Proposition 2.4.8, 3.2.11, and 3.2.12; we only prove this for irreducible Whittaker modules, but extension to other Whittaker modules is straightforward). The push-pull operation together with non-integral intertwining functors allows the induction argument to run. In the actual proof, one prove (A.2) and (A.3) first at each inductive step and use them two prove the remaining statements.

The remaining technical difficulty lies in the proof of (A.4). It requires us to find $\alpha \in \Pi_\lambda$ so that $Cs_\alpha <_{u,\lambda} C$ and (W.2) holds. If α can be chosen to be also simple in Σ^+, then (W.2) simply follows from (A.2). But there are examples where this cannot be done. The strategy then is to apply non-integral intertwining functors so that α becomes simple in both the integral Weyl group and in W, and that C is translated to a coset of smaller length so that (A.2) holds by induction assumption. (W.2) is obtained by translating (A.2) back via inverse intertwining functors. The existence of such a chain of intertwining functors is guaranteed by Lemma 2.5.1.

1.5 Outline of the dissertation

The dissertation is organized as follows. §2 is devoted to studying the structure of left W_λ-cosets and double (W_Θ, W_λ)-cosets in the Weyl group. In §3 we study intertwining functors and the U-functor, and also the effect of non-integral intertwining functors on irreducible $\mathcal{D}$-modules. §4 contains the statement and the proof of the algorithm. The character formula is established in §5. Lastly, in Appendix §A, we apply the main theorems to two small examples.

I would like to comment on some choices of inclusions (or omissions) of known results in this dissertation. The main principle is to write down details without impairing readability. §2 includes all details modulo facts on Weyl group actions on root systems, even though most results there are already known or could be left as exercises. The structures of double cosets have applications outside the current context. For example, they come up in the study of θ-stable parabolics of real reductive groups. In the first half of §3, I have included a streamlined argument for the structural results of the U-functor on certain irreducible modules. As a key part of the theory, they are included in the body of the dissertation, rather than in the appendices. Similar statements also apply to real groups with more or less the same proof. Although these results were proven in detail in [Mil], the argument there uses preliminary results that are more general than needed and are scattered in different places in *op. cit.* The only "new" part of that chapter is §3.2, even though the tools and techniques there were also known to experts. In §4, I have chosen not to present backgrounds on the story of Kazhdan-Lusztig polynomials because there already exists a vast literature on this subject. I hope the explanations in §1.4 make up for this omission.

Chapter 2

Double cosets in the Weyl group

In this section, we collect some known results on the integral root subsystem and examine the structure of double (W_Θ, W_λ)-cosets in W. Most results on here are either known or not hard. We include the proofs for completeness.

In §2.1 we define a cross-section of W/W_λ and examine the restriction of Bruhat order to each coset. §2.2 sets notations and collects known facts on $W_\Theta\backslash W$. In §2.3, we construct a cross-section $A_{\Theta,\lambda}$ of $W_\Theta\backslash W/W_\lambda$ consisting of the unique shortest elements in each double coset (Corollary 2.3.3). Next, we show in §2.4 that, if one looks at the partition of $W_\Theta\backslash W$ given by double cosets $W_\Theta\backslash W/W_\lambda$, then each block in this partition corresponds to a right coset in W_λ of a parabolic subgroup of W_λ. As mentioned in §1, the Whittaker Kazhdan-Lusztig polynomials for (W_λ, Π_λ) with respect to this parabolic subgroup describes the multiplicities of Whittaker modules indexed by right W_Θ-cosets in this double coset. Lastly, in §2.5, we prove a lemma which enables a key induction step in §4.6.

Recall that $\lambda \in \mathfrak{h}^*$, $\Sigma_\lambda = \{\alpha \in \Sigma \mid \alpha^\vee(\lambda) \in \mathbb{Z}\}$ is subsystem of integral roots, $W_\lambda = \{w \in W \mid w\lambda - \lambda \in \mathbb{Z} \cdot \Sigma\}$ is its Weyl group. Here I do not claim that Σ_λ is closed under addition in Σ, but this will not concern us. $\Sigma_\lambda^+ = \Sigma^+ \cap \Sigma_\lambda$ is the set of positive roots and $\Pi_\lambda \subseteq \Sigma_\lambda^+$ the set of simple roots. Write $\leqslant_\lambda$ for the Bruhat order on W_λ determined by Π_λ. Θ is a subset of Π, $\Sigma_\Theta \subset \Sigma$ is the subsystem generated by Θ and $W_\Theta \subset W$ is the Weyl group of Σ_Θ, identified with the subgroup of W generated by reflections of roots in Σ_Θ.

The readers can compare theses results with the examples in §A.

2.1 Left W_λ-cosets and Bruhat order

For any $u \in W$, define the set

$$\Sigma_u^+ = \{\alpha \in \Sigma^+ \mid u\alpha \in -\Sigma^+\} = \Sigma^+ \cap (-u^{-1}\Sigma^+),$$

i.e. the set of positive roots α so that $u\alpha$ is not positive. Write

$$A_\lambda = \{u \in W \mid \Sigma_u^+ \cap \Sigma_\lambda = \varnothing\}.$$

The following is well-known.

Lemma 2.1.1. *A_λ is a cross-section of W/W_λ.*

Proof. This proof is copied verbatim from Miličić's unpublished notes. Observe that

$$\begin{aligned} \Sigma_u^+ \cap \Sigma_\lambda &= \Sigma^+ \cap (-u^{-1}\Sigma^+) \cap \Sigma_\lambda && \text{(by definition of } \Sigma_u^+) \\ &= (\Sigma^+ \cap \Sigma_\lambda) \cap (-u^{-1}\Sigma^+) && \text{(rearranging terms)} \\ &= \Sigma_\lambda^+ \cap (-u^{-1}\Sigma^+) && \text{(by definition of } \Sigma_\lambda^+). \end{aligned}$$

Hence $\Sigma_u^+ \cap \Sigma_\lambda = \varnothing \iff \Sigma_\lambda^+ \subseteq u^{-1}\Sigma^+$, and

$$A_\lambda = \{u \in W \mid \Sigma_\lambda^+ \subseteq u^{-1}\Sigma^+\}. \tag{2.1.2}$$

We first show that any left W_λ-coset has a representative in A_λ. Take any $w \in W$. Then $w^{-1}\Sigma^+$ is a set of positive roots in Σ. Hence $\Sigma_\lambda \cap w^{-1}\Sigma^+$ is a set of positive roots in Σ_λ. So there is an element $v \in W_\lambda$ such that $v(\Sigma_\lambda \cap w^{-1}\Sigma^+) = \Sigma_\lambda^+$, or equivalently $\Sigma_\lambda \cap vw^{-1}\Sigma^+ = \Sigma_\lambda^+$ (because $v\Sigma_\lambda = \Sigma_\lambda$). In particular $\Sigma_\lambda^+ \subseteq vw^{-1}\Sigma^+$, and hence $(vw^{-1}) = wv^{-1} \in A_\lambda$ by the above alternative description of A_λ. As a result $w \in A_\lambda v \subseteq A_\lambda W_\lambda$. This shows $W = A_\lambda W_\lambda$, and any left W_λ-coset has a representative in A_λ.

Now suppose $u_1, u_2 \in A_\lambda$ are in the same left W_λ-coset, i.e. there is $v \in W_\lambda$ with $u_1 = u_2 v$. This implies

$$\begin{aligned} \Sigma_\lambda^+ &= \Sigma_\lambda \cap u_1^{-1}\Sigma^+ && \text{(since } u_1 \in A_\lambda \text{ and because of (2.1.2))} \\ &= \Sigma_\lambda \cap v^{-1}u_2^{-1}\Sigma^+ && \text{(since } u_1 = u_2 v) \\ &= v^{-1}(\Sigma_\lambda \cap u_2^{-1}\Sigma^+) && \text{(using } v^{-1}\Sigma_\lambda = \Sigma_\lambda \text{ and factoring } v^{-1} \text{ out)} \\ &= v^{-1}\Sigma_\lambda^+ && \text{(since } u_2 \in A_\lambda \text{ and because of (2.1.2)).} \end{aligned}$$

Since W_λ acts simply transitively on the set of sets of positive roots of Σ_λ, we have $v = 1$ and $u_1 = u_2$. Thus A_λ is a cross-section of W/W_λ. □

Σ_λ, W_λ and A_λ satisfy the following elementary properties. The proof is an easy exercise.

Lemma 2.1.3. *Let β be a simple root and let $u \in W$. Write s_β for the reflection of β.*

(a) $u\Sigma_\lambda = \Sigma_{u\lambda}$;

(b) *if* $u \in A_\lambda$, $u\Sigma_\lambda^+ = \Sigma_{u\lambda}^+$*;*

(c) *if* $u \in A_\lambda$, $u\Pi_\lambda = \Pi_{u\lambda}$*;*

(d) $uW_\lambda u^{-1} = W_{u\lambda}$*;*

(e) *if* $s_\beta \in A_\lambda$ *and* $u \in A_\lambda$, $us_\beta \in A_{s_\beta\lambda}$*.*

(f) $s_\beta \in A_\lambda$ *if and only if* $\beta \in \Pi - \Pi_\lambda$*.*

Proof. (a): for any $\alpha \in \Sigma_\lambda$, $(u\alpha)^\vee(u\lambda) = \alpha^\vee(u^{-1}u\lambda) = \alpha^\vee(\lambda) \in \mathbb{Z}$. Hence $u\alpha \in \Sigma_{u\lambda}$ and $u\Sigma_\lambda \subseteq \Sigma_{u\lambda}$ by the definition of $\Sigma_{u\lambda}$. Since both sets have the same size, equality holds.

(b): from (2.1.2), we know $u\Sigma_\lambda^+ \subseteq \Sigma^+$. Hence

$$u\Sigma_\lambda^+ = u\Sigma_\lambda \cap \Sigma^+ = \Sigma_{u\lambda} \cap \Sigma^+ = \Sigma_{u\lambda}^+.$$

(c): elements in Π_λ and $\Pi_{u\lambda}$ can be characterized by not being sums of other elements of Σ_λ^+ and $\Sigma_{u\lambda}^+$, respectively. Since $u : \Sigma_\lambda^+ \to \Sigma_{u\lambda}^+$ commutes with sums, it must send Π_λ to $\Pi_{u\lambda}$.

(d): for any $w \in W_\lambda$,

$$(uwu^{-1})u\lambda - u\lambda = u(w\lambda - \lambda) \in u(\mathbb{Z} \cdot \Sigma) = \mathbb{Z} \cdot \Sigma.$$

Hence $uW_\lambda u^{-1} \subseteq W_{u\lambda}$ by definition of $W_{u\lambda}$. Since both sides have the same size, equality holds.

(e): observe

$$\begin{aligned}
\Sigma_{us_\beta}^+ \cap \Sigma_{s_\beta\lambda} &= (\Sigma^+ \cap (-(us_\beta)^{-1}\Sigma^+)) \cap \Sigma_{s_\beta\lambda} && \text{(by definition of } \Sigma_{us_\beta}^+) \\
&= (\Sigma^+ \cap \Sigma_{s_\beta\lambda}) \cap (-(us_\beta)^{-1}\Sigma^+) && \text{(rearranging terms)} \\
&= \Sigma_{s_\beta\lambda}^+ \cap (-(us_\beta)^{-1}\Sigma^+) && \text{(by definition of } \Sigma_{s_\beta\lambda}^+) \\
&= s_\beta\Sigma_\lambda^+ \cap (-(us_\beta)^{-1}\Sigma^+) && \text{(by part (b)).}
\end{aligned}$$

Hence

$$\begin{aligned}
us_\beta \in A_{s_\beta\lambda} &\iff \Sigma_{us_\beta}^+ \cap \Sigma_{s_\beta\lambda} = \varnothing && \text{(by definition of } A_{s_\beta\lambda}) \\
&\iff s_\beta\Sigma_\lambda^+ \cap (-(us_\beta)^{-1}\Sigma^+) = \varnothing && \text{(by the above observation)} \\
&\iff s_\beta\Sigma_\lambda^+ \subseteq (us_\beta)^{-1}\Sigma^+ = s_\beta u^{-1}\Sigma^+ \\
&\iff u\Sigma_\lambda^+ \subseteq \Sigma^+ && \text{(multiplying both sides by } us_\beta) \\
&\iff u \in A_\lambda && \text{(by (2.1.2))}
\end{aligned}$$

which is true by assumption.

(f): if $s_\beta \in A_\lambda$, then since A_λ is a cross-section of W/W_λ and 1 is already in W_λ, we must have $s_\beta \notin W_\lambda$. Hence

$$-\beta^\vee(\lambda)\beta = (\lambda - \beta^\vee(\lambda)\beta) - \lambda = s_\beta\lambda - \lambda \notin \mathbb{Z} \cdot \Sigma,$$

and $\beta^\vee(\lambda) \notin \mathbb{Z}$. Therefore $\beta \notin \Pi_\lambda$. On the other direction, suppose $\beta \notin \Pi_\lambda$. Since the only positive root moved out of Σ^+ by s_β is β, and β is not in Σ_λ, we see that $s_\beta \Sigma^+_\lambda \subseteq \Sigma^+$. This implies $s_\beta \in A_\lambda$ by (2.1.2). □

In particular, (c) and (d) imply that conjugation by $u \in A_\lambda$ sends simple reflections in W_λ to simple reflections in $W_{u\lambda}$. This implies:

Corollary 2.1.4. *Let $u \in A_\lambda$. Then conjugation by u is an isomorphism of posets*

$$(W_\lambda, \leqslant_\lambda) \xrightarrow{\sim} (W_{u\lambda}, \leqslant_{u\lambda}).$$

We want to show that A_λ consists of unique shortest elements in left cosets. We in fact have a stronger statement: left multiplication by an element in A_λ is a map from W_λ to W that preserves the Bruhat orders.

Lemma 2.1.5. *Let w, $s_\alpha \in W$ with $\alpha \in \Sigma^+$. Let $u \in W$ such that $u\alpha \in \Sigma^+$. Let μ be a regular antidominant integral weight. Then*

$$us_\alpha w < uw \iff us_\alpha w\mu < uw\mu.$$

Here the left hand side is the Bruhat order, and the right hand side means that $uw\mu - us_\alpha w\mu$ is nonzero and is a non-negative sum of simple roots.

Proof. We rewrite

$$us_\alpha w\mu = uw\mu - \alpha^\vee(w\mu)u\alpha = uw\mu - (w^{-1}\alpha)^\vee(\mu)u\alpha.$$

Hence

$$\begin{aligned}
us_\alpha w\mu < uw\mu &\iff (w^{-1}\alpha)^\vee(\mu)u\alpha > 0 && \\
&\iff (w^{-1}\alpha)^\vee(\mu) > 0 && \text{(because } u\alpha \in \Sigma^+\text{)} \\
&\iff w^{-1}\alpha \notin \Sigma^+ && \text{(because } \mu \text{ is antidominant regular)} \\
&\iff (uw)^{-1}(u\alpha) \notin \Sigma^+ && \\
&\iff \ell(s_{u\alpha}uw) < \ell(uw) && \text{(by [Bou02, VI.1.6 Proposition 17(ii)])} \\
&\iff s_{u\alpha}uw < uw && \text{(by definition of Bruhat order).}
\end{aligned}$$

Finally, observe that $s_{u\alpha}uw = us_\alpha u^{-1}uw = us_\alpha w$ by Lemma 2.1.3(4). Thus $us_\alpha w\mu < uw\mu$ is equivalent to $us_\alpha w < uw$, as desired. □

Lemma 2.1.6. *Let $w, s_\alpha \in W_\lambda$ with $\alpha \in \Sigma_\lambda^+$, and let $u \in A_\lambda$. Suppose $s_\alpha w <_\lambda w$. Then $us_\alpha w < uw$.*

Proof. Consider the projection $\mathfrak{h}^* \twoheadrightarrow \operatorname{span}\Sigma_\lambda$ along the subspace $\bigcap_{\alpha\in\Sigma_\lambda}\ker\alpha$. For an element $\mu \in \mathfrak{h}^*$, we write $\bar{\mu}$ for its image under this projection.

By the preceding lemma 2.1.5, an inequality in W with respect to Bruhat order can be checked by a regular antidominant integral weight. That is, if μ is such a weight in $\mathfrak{h}^*$, then $us_\alpha w < uw$ if and only if $us_\alpha w\mu < uw\mu$. Similarly, $s_\alpha w <_\lambda w$ if and only if $s_\alpha w\bar{\mu} <_\lambda w\bar{\mu}$.

Therefore, if we write $\nu = \mu - \bar{\mu}$,

$$\begin{aligned} s_\alpha w <_\lambda w &\iff s_\alpha w\bar{\mu} <_\lambda w\bar{\mu} \\ &\iff s_\alpha w\bar{\mu} + \sum_{\alpha_i\in\Pi_\lambda} a_i\alpha_i = w\bar{\mu} \text{ for some } a_i \in \mathbb{Z}_{\geqslant 0} \text{ not all zero} \\ &\iff s_\alpha w\bar{\mu} + \nu + \sum_{\alpha_i\in\Pi_\lambda} a_i\alpha_i = w\bar{\mu} + \nu \text{ for some } a_i \in \mathbb{Z}_{\geqslant 0} \text{ not all zero} \\ &\iff s_\alpha w\mu + \sum_{\alpha_i\in\Pi_\lambda} a_i\alpha_i = w\mu \text{ for some } a_i \in \mathbb{Z}_{\geqslant 0} \text{ not all zero} \end{aligned}$$

where the last step is because ν is annihilated by all coroots in $\Sigma_\lambda^\vee$. Applying u to both sides we get

$$us_\alpha w\mu + \sum_{\alpha_i\in\Pi_\lambda} a_i u\alpha_i = uw\mu \text{ for some } a_i \geqslant 0 \text{ not all zero.}$$

Since each $u\alpha_i$ is positive ($u\alpha_i \in u\Sigma_\lambda^+ \subseteq \Sigma^+$), we have $us_\alpha w\mu < uw\mu$. Thus $us_\alpha w < uw$ as desired. □

Corollary 2.1.7. *Let $v, w \in W_\lambda$ and $v \leqslant_\lambda w$. Then for any $u \in A_\lambda$, $uv \leqslant uw$.*

Proof. If equality holds, then the statement is trivial. Otherwise, by the definition of Bruhat order, there exist $\alpha_1, \dots, \alpha_k \in \Sigma_\lambda^+$ such that

$$v = s_{\alpha_k}\cdots s_{\alpha_1}w <_\lambda \cdots <_\lambda s_{\alpha_1}w <_\lambda w.$$

Apply Lemma 2.1.6 to each inequality, we see

$$uv = us_{\alpha_k}\cdots s_{\alpha_1}w < \cdots < us_{\alpha_1}w < uw$$

as desired. □

In particular,

Corollary 2.1.8. *For any $u \in A_\lambda$, u is the unique shortest element in uW_λ with respect to the restriction of Bruhat order to uA_λ.*

Remark 2.1.9. Note that the proof of 2.1.7 only uses the fact that Σ_λ is a subsystem of Σ (again I do not claim that Σ_λ is closed under addition in Σ) and Σ_λ^+ is defined as the intersection $\Sigma_\lambda \cap \Sigma^+$. Therefore the same argument can be applied to other root subsystems such as Σ_Θ. The same holds for the next lemma 2.1.10. This will be used in §2.2.

The next lemma is analogous to a similar statement for parabolic subgroups (Lemma 2.2.3), which we will need in a few occasions. The proof is a standard argument using the lifting property [BB05, 2.2.7].

Lemma 2.1.10. *Let $\alpha \in \Pi$, and $u \in A_\lambda$. Then either $s_\alpha u \in A_\lambda$, or $s_\alpha u \in uW_\lambda$.*

Proof. Suppose $s_\alpha u \notin uW_\lambda$. Then $s_\alpha u$ is in a different left W_λ-coset, i.e. $s_\alpha u = rv \in rW_\lambda$ for some $v \in W_\lambda$ and $r \in A_\lambda$ with $r \neq u$. So there exists some $v \in W_\lambda$ such that $s_\alpha u = rv$. We need to show that $v = 1$.

Write $w_1 \lhd w_2$ when $w_1 < w_2$ and $\ell(w_1) = \ell(w_2) - 1$. From the relation $s_\alpha u = rv$, either $rv \lhd u$ or $rv \rhd u$. Also $s_\alpha uv^{-1} = r$, so either $r \lhd uv^{-1}$ or $r \rhd uv^{-1}$. From Corollary 2.1.8, we also know $r \leqslant rv$ and $u \leqslant uv^{-1}$. We have the following four possibilities.

(a) $\begin{array}{ccc} r & \rhd & uv^{-1} \\ \wedge\!\!| & & \vee\!\!| \\ rv & \lhd & u \end{array}$ is impossible since it implies $u > u$.

(b) $\begin{array}{ccc} r & \rhd & uv^{-1} \\ \wedge\!\!| & & \vee\!\!| \\ rv & \rhd & u \end{array}$. If $rv > r$, then from $rv > r \rhd uv^{-1} \geqslant u$ we see that $\ell(rv) \geqslant \ell(u) + 2$, which violates $rv \rhd u$. Therefore we must have $rv = r$ and hence $v = 1$.

(c) $\begin{array}{ccc} r & \lhd & uv^{-1} \\ \wedge\!\!| & & \vee\!\!| \\ rv & \lhd & u \end{array}$. Same argument as in (b) shows that $v = 1$.

(d) $\begin{array}{ccc} r & \lhd & uv^{-1} \\ \wedge\!\!| & & \vee\!\!| \\ rv & \rhd & u \end{array}$. Let $k = \ell(uv^{-1}) - \ell(u)$. Then

$$\begin{aligned} \ell(rv) &\geqslant \ell(r) \\ &= \ell(uv^{-1}) - 1 \\ &= \ell(u) + k - 1 \\ &= \ell(rv) - 1 + k - 1 \\ &= \ell(rv) + k - 2 \end{aligned}$$

and $0 \leqslant k \leqslant 2$. If $k = 2$, then $\ell(r) = \ell(rv)$ and $v = 1$. If $k = 0$, then $\ell(u) = \ell(uv^{-1})$ and $v = 1$. Suppose $k = 1$. Applying the lifting property twice, we see $r \leqslant u$ and $u \leqslant r$. So $r = u$, contradicting our assumption for r. Therefore we must have $v = 1$.

Thus $v = 1$ in all cases, as desired. □

2.2 Notations and preliminaries on $W_\Theta\backslash W$

We recall some well-known facts of right W_Θ-cosets and partial orders. Details these facts can be found in [Mil] in the chapter on generalized Verma modules (see also [BB05, §2.5] for proofs of these restuls for left W_Θ-cosets).

A similar proof as Corollary 2.1.7 shows that the set

$$^{\Theta}W = \{w \in W \mid w^{-1}\Theta \subseteq -\Sigma^+\}$$

is a cross-section of $W_\Theta\backslash W$ consisting of the longest elements in each coset. Write $w_\Theta \in W_\Theta$ for the longest element. Then

$$w_\Theta{}^{\Theta}W = \{w \in W \mid w^{-1}\Theta \subseteq \Sigma^+\} \tag{2.2.1}$$

is a cross-section consisting of the shortest elements in each coset. For a right W_Θ-coset C, we write w^C for the corresponding element in $^{\Theta}W$. The restriction of Bruhat order on the set $^{\Theta}W$ defines a partial order $\leqslant$ on $W_\Theta\backslash W$. We will use "the length of C" and $\ell(C)$ to refer to the length of the element w^C. If $\Theta(u,\lambda)$ is a subset of Π_λ defining a parabolic subgroup $W_{\lambda,\Theta(u,\lambda)} \subseteq W_\lambda$, we write "$\leqslant_{u,\lambda}$" for the partial order on $W_{\lambda,\Theta(u,\lambda)}\backslash W_\lambda$.

The following facts will be used throughout the rest of this chapter.

Lemma 2.2.2. *Any element in W_Θ permutes positive roots outside Σ^+_Θ, that is, it permutes the set $\Sigma^+ - \Sigma^+_\Theta$.*

Proof. It suffices to prove it for a simple reflection $s_\beta \in W_\Theta$. s_β permutes $\Sigma^+ - \{\pm\beta\}$ and also permutes $\Sigma - \Sigma_\Theta$, whence it permutes $(\Sigma^+ - \{\pm\beta\}) \cap (\Sigma - \Sigma_\Theta)$ which equals $\Sigma^+ - \Sigma^+_\Theta$. □

Lemma 2.2.3. *Let C be a right W_Θ-coset and $\alpha \in \Pi$. Then exactly one of the following happens.*

(a) $Cs_\alpha > C$. *In this case $w^{Cs_\alpha} = w^C s_\alpha$, and for any $w \in C$, $ws_\alpha > w$.*

(b) $Cs_\alpha = C$.

(c) $Cs_\alpha < C$. *In this case $w^{Cs_\alpha} = w^C s_\alpha$, and for any $w \in C$, $ws_\alpha < w$.*

Moreover, the identity coset W_Θ is the only right W_Θ-coset C such that $Cs_\alpha \geqslant C$ for all $\alpha \in \Pi$.

Proof. With minor modifications, the results in §2.1 can be translated to the case where we replace Σ_λ by Σ_Θ, left W_λ-cosets by right W_Θ-cosets, and A_λ by ${}^\Theta W$. Under these replacements, Lemmas 2.1.7 and 2.1.10 say the following:

(i) Let $v, w \in W_\Theta$ and $v \leqslant w$. Then for any $C \in W_\Theta \backslash W$, $vw^C \geqslant ww^C$.

(ii) Let $\alpha \in \Pi$ and $C \in W_\Theta \backslash W$. Then either $w^C s_\alpha = w^{Cs_\alpha}$ or $w^C s_\alpha \in C$.

The second case of (ii) ($w^C s_\alpha \in C$) corresponds to (b). Suppose we are in the first case of (ii), that is $w^C s_\alpha = w^{Cs_\alpha}$. Then $Cs_\alpha \neq C$, otherwise $w^C s_\alpha = w^{Cs_\alpha} = w^C$ is impossible. Suppose $Cs_\alpha > C$, i.e. $w^{Cs_\alpha} > w^C$. We want to show that $ws_\alpha > w$ for any $w \in C$.

Since $C = W_\Theta w^C$, we can write $w = vw^C$ for some unique $v \in W_\Theta$. We will do induction on $\ell(v)$. For the base case $\ell(v) = 1$, $v = s_\beta$ is simple. (i) implies $s_\beta w^{Cs_\alpha} > w^{Cs_\alpha}$ and $s_\beta w^C > w^C$. Combined with the assumption $w^{Cs_\alpha} > w^C$, we obtain

$$\begin{array}{ccc} s_\beta w^{Cs_\alpha} & < & w^{Cs_\alpha} \\ & & \vee \\ s_\beta w^C & < & w^C \end{array}.$$

If $s_\beta w^C s_\alpha < s_\beta w^C$, then the chain of inequalities

$$\begin{array}{ccc} s_\beta w^{Cs_\alpha} & & w^{Cs_\alpha} \\ \wedge & & \vee \\ s_\beta w^C & < & w^C \end{array}$$

would imply that $s_\beta w^{Cs_\alpha}$ and w^{Cs_α} have length difference $\geqslant 3$, which is impossible since their lengths only differ by 1. Therefore $ws_\alpha = s_\beta w^C s_\alpha > s_\beta w^C = w$. This establishes the base case.

Now suppose $v = s_\beta r > r$ for some $s_\beta, r \in W_\Theta$, with s_β simple. Induction hypothesis says $rw^{Cs_\alpha} > rw^C$. Invoking (i) again, we obtain the following inequalities

$$\begin{array}{ccc} s_\beta rw^{Cs_\alpha} & < & rw^{Cs_\alpha} \\ & & \vee \\ s_\beta rw^C & < & rw^C \end{array}.$$

Arguing similarly as in the base case, it is impossible to have $s_\beta rw^{Cs_\alpha} < s_\beta rw^C$. Therefore $ws_\alpha = s_\beta rw^{Cs_\alpha} > s_\beta rw^C = w$. This proves the additional claim in case (a). An identical argument establishes the claim in case (c).

It remains to prove the last statement. Suppose C satisfies $Cs_\alpha \geqslant C$ for all $\alpha \in \Pi$. Let $w \in C$ be an element of minimal length. If $w \neq 1$, then $w > 1$, and there is a simple reflection s_α with $ws_\alpha < w$. By minimality of w, ws_α is in a different coset. This forces us to be in case (c), which contradicts $Cs_\alpha \geqslant C$. So $w = 1$ and $C = W_\Theta$. □

This immediately implies

Corollary 2.2.4. *Let* C, D *be two right* W_Θ*-cosets. Let* $v \in D$, $w \in C$. *If* $v \leqslant w$, *then* $D \leqslant C$.

Proof. We can choose simple roots α_i's so that

$$w = vs_{\alpha_k} \cdots s_{\alpha_2} s_{\alpha_1} \geqslant vs_{\alpha_k} \cdots s_{\alpha_2} \geqslant \cdots \geqslant v.$$

By the lemma, this implies

$$C \geqslant Cs_{\alpha_1} \geqslant \cdots \geqslant Cs_{\alpha_1} \cdots s_{\alpha_k} = D.$$ □

2.3 A cross-section of $W_\Theta \backslash W / W_\lambda$

Define the set

$$A_{\Theta,\lambda} = A_\lambda \cap (w_\Theta {}^\Theta W) = \{u \in W \mid \Sigma_\lambda^+ \subseteq u^{-1}\Sigma^+,\ \Theta \subseteq u\Sigma^+\}.$$

We will show (in Corollary 2.3.3) that this is a cross-section of $W_\Theta \backslash W / W_\lambda$ consisting of the unique shortest elements in each double coset. Later results, as well as the main theorem of the thesis, will often be formulated using this set.

We first show that elements in A_λ are concentrated on the lowest W_Θ-layers in the double cosets.

Lemma 2.3.1. *Let* $u, r \in A_\lambda$. *Suppose* u *and* r *are in the same* (W_Θ, W_λ)*-coset. Then* u *and* r *are contained in the same right* W_Θ*-coset.*

Proof. The case $u = r$ is trivial. Assume $u \neq r$. By assumption, $r = wuv^{-1}$ for some $w \in W_\Theta$ and $v \in W_\lambda$. We will do induction on $\ell(w)$.

Consider the case $\ell(w) = 1$. Then $w = s_\alpha$ for some $\alpha \in \Theta$, and $s_\alpha u = rv \in rW_\lambda$. By Lemma 2.1.10, $s_\alpha u$ is either in A_λ or in uW_λ. But the second case is impossible because uW_λ is disjoint from rW_λ. So $s_\alpha u = rv \in A_\lambda \cap rW_\lambda$. Since A_λ is a cross-section of W/W_λ, this intersection is equal to r. Hence $v = 1$, and $r = wu \in W_\Theta u$ which is in the same right W_Θ-coset as u.

Consider $\ell(w) = k > 1$. Write $w = s_\alpha w' > w'$ for some $s_\alpha, w' \in W_\Theta$. Then $r = wuv^{-1}$ can be rewritten as $w'u = (s_\alpha r)v$. We have two possibilities.

(a) $s_\alpha r \in rW_\lambda$, i.e. $s_\alpha r = rv'$ for some $v' \in W_\lambda$. So the equality $w'u = (s_\alpha r)v$ becomes $w'u(v'v)^{-1} = r$ with $w' \in W_\Theta$ and $v'v \in W_\lambda$. Since $\ell(w') \leqslant k-1$, by the induction assumption, u and r are in the same right W_Θ-coset.

(b) $s_\alpha r \notin rW_\lambda$. Then by Lemma 2.1.10, $s_\alpha r \in A_\lambda$. From the equation $w'uv^{-1} = s_\alpha r$, $\ell(w') \leqslant k-1$ and the induction assumption, we see that u and $s_\alpha r$ are in the same right W_Θ-coset. Since $s_\alpha r$ and r are in the same right W_Θ-coset, so are u and r. $\square$

Proposition 2.3.2. *Consider any double coset $W_\Theta w W_\lambda$ in W.*

(a) $W_\Theta w W_\lambda$ contains a unique smallest right W_Θ-coset C, in the sense that $C \leqslant C'$ for any $C' \in W_\Theta \backslash W_\Theta w W_\lambda$.

(b) $A_\lambda \cap (W_\Theta w W_\lambda) \subseteq C$.

(c) The unique shortest element in C is in A_λ.

Proof. By the preceding lemma 2.3.1, there exists a right W_Θ-coset C, contained in $W_\Theta w W_\lambda$, such that $A_\lambda \cap (W_\Theta w W_\lambda) \subseteq C$. Let y be the unique shortest element in C. y belongs to some left W_λ coset, say to uW_λ for some $u \in A_\lambda$. Then $u \leqslant y$ by since u is shortest in uW_λ (Corollary 2.1.8). If $y \neq u$, we will have $u < y$, and hence by minimality of y, u is in a different right W_Θ-coset than y, contradicting the construction of C. Hence we must have $y = u$, i.e. the unique shortest element in C is in A_λ. Lastly, for any other right W_Θ-coset C' in $W_\Theta w W_\lambda$, let y' be its unique shortest element. y' is contained in one of the left W_λ-cosets, say $y' \in u'W_\lambda$ for some $u' \in A_\lambda$. Then $u' \leqslant y'$ by Corollary 2.1.8. Also $u' \neq y'$ (otherwise $C' \ni y' = u' \in C$ which would imply $C = C'$). Hence $u' < y'$. Therefore $C < C'$ by Corollary 2.2.4. Thus C is the unique smallest right W_Θ-coset in $W_\Theta w W_\lambda$. $\square$

The above proof is based on the fact that A_λ consists of shortest elements in left W_λ-cosets. Combined with the fact that $w_\Theta {}^\Theta W$ consists of shortest elements in right W_Θ-cosets, we obtain:

Corollary 2.3.3. *$A_{\Theta,\lambda} := A_\lambda \cap (w_\Theta {}^\Theta W)$ is a cross-section of $W_\Theta \backslash W / W_\lambda$ consisting of the unique shortest elements in each double coset. For each $u \in A_{\Theta,\lambda}$, $W_\Theta u$ is the unique smallest right W_Θ-coset in $W_\Theta u W_\lambda$.*

Proof. Take any double coset $W_\Theta w W_\lambda$. By Proposition 2.3.2(c), if we take the shortest element u in the smallest right W_Θ-coset in this double coset, then $u \in A_\lambda$. Hence $u \in A_\lambda \cap (w_\Theta {}^\Theta W)$. Any other element in this smallest right W_Θ-coset is not in $A_\lambda \cap (w_\Theta {}^\Theta W)$ because they are not in $w_\Theta {}^\Theta W$. On the other hand, by 2.3.2(b), any other right W_Θ-coset in $W_\Theta w W_\lambda$ has empty intersection with A_λ. Therefore u is the unique element in $A_\lambda \cap (w_\Theta {}^\Theta W) \cap W_\Theta w W_\lambda$. This shows that $A_\lambda \cap (w_\Theta {}^\Theta W)$ is a cross-section. $\square$

2.4 Integral models

By results of the previous section, for each double coset $W_\Theta u W_\lambda$ one can choose u to be in $A_{\Theta,\lambda}$. Then uW_λ is contained in $W_\Theta u W_\lambda$ and it intersects with different right W_Θ-cosets. It will turn out that these intersections produce a parabolic subgroup in W_λ, and the Whittaker Kazhdan-Lusztig polynomials for W_λ that arise determine the coefficients in the character formula.

The first task is to show that there is actually a parabolic subgroup related to the intersections.

Lemma 2.4.1. *Let $u \in A_{\Theta,\lambda}$. Then $\Sigma_\Theta \cap \Pi_{u\lambda}$ is a set of simple roots for the root system $\Sigma_\Theta \cap \Sigma_{u\lambda}$.*

Proof. Let $\beta \in \Sigma_\Theta \cap \Sigma_{u\lambda}$. Write β as a $\mathbb{Z}_{\geqslant 0}$-linear combination in terms of reflections of roots in $\Pi_{u\lambda}$. If one of the summands is from $\Pi_{u\lambda} - \Sigma_\Theta$, then writing β as a sum of reflections of roots in Π, there is a summand that comes from $\Pi - \Theta$. This implies $\beta \notin \Sigma_\Theta$, a contradiction. Hence β is a sum of reflections of roots from $\Sigma_\Theta \cap \Pi_{u\lambda}$. Therefore $\Sigma_\Theta \cap \Pi_{u\lambda}$ spans $\Sigma_\Theta \cap \Sigma_{u\lambda}$. Since $\Sigma_\Theta \cap \Pi_{u\lambda}$ is a subset of simple roots in $\Sigma_{u\lambda}$, roots in $\Sigma_\Theta \cap \Pi_{u\lambda}$ remain simple in $\Sigma_\Theta \cap \Sigma_{u\lambda}$. Thus $\Sigma_\Theta \cap \Pi_{u\lambda}$ is a set of simple roots for $\Sigma_\Theta \cap \Sigma_{u\lambda}$. □

Write $W_{u\lambda,\Sigma_\Theta\cap\Pi_{u\lambda}}$ for the parabolic subgroup of $W_{u\lambda}$ corresponding to $\Sigma_\Theta \cap \Pi_{u\lambda}$. Then $W_{u\lambda,\Sigma_\Theta\cap\Pi_{u\lambda}}$ is the Weyl group of $\Sigma_\Theta \cap \Sigma_{u\lambda}$ and is a subgroup of $W_\Theta \cap W_{u\lambda}$.

Proposition 2.4.2. *For any $u \in A_{\Theta,\lambda}$, $W_\Theta \cap W_{u\lambda} = W_{u\lambda,\Sigma_\Theta\cap\Pi_{u\lambda}}$. In particular, $W_\Theta \cap W_{u\lambda}$ is a parabolic subgroup of $W_{u\lambda}$.*

Proof. $W_{u\lambda,\Sigma_\Theta\cap\Pi_{u\lambda}}$ is certainly contained in $W_\Theta \cap W_{u\lambda}$. Let $w \in W_\Theta \cap W_{u\lambda}$. Being in W_Θ, w permutes roots in Σ_Θ; being in $W_{u\lambda}$, w permutes roots in $\Sigma_{u\lambda}$. Hence w permutes roots in $\Sigma_\Theta \cap \Sigma_{u\lambda}$, and it sends the set $\Sigma^+ \cap (\Sigma_\Theta \cap \Sigma_{u\lambda})$ of positive roots in $\Sigma_\Theta \cap \Sigma_{u\lambda}$ to another set of positive roots $w\Sigma^+ \cap (\Sigma_\Theta \cap \Sigma_{u\lambda})$. Since $W_{u\lambda,\Sigma_\Theta\cap\Pi_{u\lambda}}$ is the Weyl group of $\Sigma_\Theta \cap \Sigma_{u\lambda}$, there exists a unique element $v \in W_{u\lambda,\Sigma_\Theta\cap\Pi_{u\lambda}}$ that sends $w\Sigma^+ \cap (\Sigma_\Theta \cap \Sigma_{u\lambda})$ back to $\Sigma^+ \cap (\Sigma_\Theta \cap \Sigma_{u\lambda})$. Hence vw permutes $\Sigma^+ \cap (\Sigma_\Theta \cap \Sigma_{u\lambda}) = \Sigma^+_{u\lambda} \cap \Sigma^+_\Theta$. On the other hand, since $vw \in W_\Theta$, by Lemma 2.2.2 it permutes $\Sigma^+ - \Sigma^+_\Theta$; vw is also in $W_{u\lambda}$, so it permutes $\Sigma_{u\lambda}$. Hence, it permutes $(\Sigma^+ - \Sigma^+_\Theta) \cap \Sigma_{u\lambda} = \Sigma^+_{u\lambda} - \Sigma^+_\Theta$. As a result, vw permutes

$$(\Sigma^+_{u\lambda} \cap \Sigma^+_\Theta) \cup (\Sigma^+_{u\lambda} - \Sigma^+_\Theta) = \Sigma^+_{u\lambda}.$$

Since $W_{u\lambda}$ acts simply transitively on the set of sets of positive roots in $\Sigma_{u\lambda}$, we must have $vw = 1$. Therefore $w = v^{-1} \in W_{u\lambda,\Sigma_\Theta\cap\Pi_{u\lambda}}$. Thus $W_\Theta \cap W_{u\lambda} = W_{u\lambda,\Sigma_\Theta\cap\Pi_{u\lambda}}$, as desired. □

For $u \in A_{\Theta,\lambda}$, write

$$\Theta(u,\lambda) = u^{-1}(\Sigma_\Theta \cap \Pi_{u\lambda}) = u^{-1}\Sigma_\Theta \cap \Pi_\lambda.$$

Since $\Sigma_\Theta \cap \Pi_{u\lambda}$ is a subset of simple roots in $\Sigma_{u\lambda}$, $\Theta(u,\lambda)$ is a subset of simple roots in $u^{-1}\Sigma_{u\lambda} = \Sigma_\lambda$. Write $W_{\lambda,\Theta(u,\lambda)}$ for the parabolic subgroup of W_λ corresponding to $\Theta(u,\lambda)$.

Proposition 2.4.3. *Let $u \in A_{\Theta,\lambda}$. The left-multiplication-by-u map*

$$W_\lambda \xrightarrow{\sim} uW_\lambda$$

induces bijections

$$\begin{array}{ccccc} W_{\lambda,\Theta(u,\lambda)}\backslash W_\lambda & \xrightarrow{\sim} & \{C \cap uW_\lambda \mid C \in W_\Theta\backslash W_\Theta u W_\lambda\} & \xrightarrow{\sim} & W_\Theta\backslash W_\Theta u W_\lambda \\ W_{\lambda,\Theta(u,\lambda)}\nu & \mapsto & uW_{\lambda,\Theta(u,\lambda)}\nu = W_\Theta u\nu \cap uW_\lambda & \mapsto & W_\Theta u\nu. \end{array}$$

Moreover, this map preserves the partial orders on cosets: if $C', D' \in W_{\lambda,\Theta(u,\lambda)}\backslash W_\lambda$ are sent to $C \cap uW_\lambda$ and $D \cap uW_\lambda$, respectively, then $D' \leqslant_{u,\lambda} C'$ implies $D \leqslant C$.

Proof. Consider the smallest right W_Θ-coset $W_\Theta u$ of $W_\Theta u W_\lambda$. Then

$$\begin{aligned} W_\Theta u \cap uW_\lambda &= (W_\Theta \cap uW_\lambda u^{-1})u && \text{(factoring out } u) \\ &= (W_\Theta \cap W_{u\lambda})u && \text{(by Lemma 2.1.3(d))} \\ &= W_{u\lambda,\Sigma_\Theta\cap\Pi_{u\lambda}}u && \text{(by Proposition 2.4.2)} \\ &= W_{u\lambda,u\Theta(u,\lambda)}u && \text{(by definition of } \Theta(u,\lambda)) \\ &= (uW_{\lambda,\Theta(u,\lambda)}u^{-1})u && \text{(by Lemma 2.1.3(d))} \\ &= uW_{\lambda,\Theta(u,\lambda)}. \end{aligned}$$

Hence left multiplication by u sends the identity coset $W_{\lambda,\Theta(u,\lambda)}1$ to $W_\Theta u \cap uW_\lambda$. Since left multiplication by u commutes with right multiplication by elements of W_λ, it sends right $W_{\lambda,\Theta(u,\lambda)}$-cosets in W_λ to right W_λ-translates of $W_\Theta u \cap uW_\lambda$, which gives us $C \cap uW_\lambda$ for various right W_Θ-cosets C in $W_\Theta u W_\lambda$. Moreover, any right W_Θ-coset C in $W_\Theta u W_\lambda$ is obtained as a right W_λ-translation of $W_\Theta u$, hence the intersection $C \cap uW_\lambda$ is necessarily the image of a right $W_{\lambda,\Theta(u,\lambda)}$-coset.

To show that this map is order preserving, take two right $W_{\lambda,\Theta(u,\lambda)}$-cosets C' and D' such that $D' \leqslant_{u,\lambda} C'$. This means that the $\leqslant_\lambda$-longest elements $\nu^{D'}$, $\nu^{C'}$ of D' and C' satisfy $\nu^{D'} \leqslant_\lambda \nu^{C'}$. Since left multiplication by u preserves Bruhat orders (Corollary 2.1.7), $D \ni u\nu^{D'} \leqslant u\nu^{C'} \in C$. Therefore $D \leqslant C$ by Corollary 2.2.4. ∎

Corollary 2.4.4. *As u ranges over $A_{\Theta,\lambda}$, left multiplication by $W_\Theta u$ defines a bijection*

$$\mathrm{ind}_\lambda : \bigcup_{u \in A_{\Theta,\lambda}} W_{\lambda,\Theta(u,\lambda)}\backslash W_\lambda \xrightarrow{\sim} W_\Theta\backslash W, \quad W_{\lambda,\Theta(u,\lambda)}\nu \mapsto W_\Theta u\nu$$

which is order-preserving when restricted to each $W_{\lambda,\Theta(u,\lambda)}\backslash W_\lambda$ *and commutes with right multiplication by* W_λ*. The image of* $W_{\lambda,\Theta(u,\lambda)}\backslash W_\lambda$ *equals* $W_\Theta\backslash W_\Theta u W_\lambda$*.*

Notation 2.4.5. We write

$$(-)|_\lambda : W_\Theta\backslash W \to \bigcup_{u\in A_{\Theta,\lambda}} W_{\lambda,\Theta(u,\lambda)}\backslash W_\lambda$$

for the inverse map to ind_λ. If C and D are both in $W_\Theta u W_\lambda$ (so that they are both sent to $W_{\lambda,\Theta(u,\lambda)}\backslash W_\lambda$), we will write $C \leqslant_{u,\lambda} D$ for $C|_\lambda \leqslant_{u,\lambda} D|_\lambda$, so that

$$C \leqslant_{u,\lambda} D \text{ is equivalent to } C, D \in W_\Theta\backslash W_\Theta u W_\lambda \text{ and } C|_\lambda \leqslant_{u,\lambda} D|_\lambda.$$

By abuse of notation, we will write $C \nleqslant_{u,\lambda} D$ if C and D are not in the same (W_Θ, W_λ)-coset, or if they are in the same coset $W_\Theta u W_\lambda$ but $C|_\lambda \nleqslant_{u,\lambda} D|_\lambda$.

The map $(-)|_\lambda$ plays an important role towards our goal. As explained in the introduction, standard and irreducible Whittaker modules in $\mathcal{N}_{\theta,\eta}$ are parameterized by $W_\Theta\backslash W$, but compared to the integral case, $\mathcal{N}_{\theta,\eta}$ is divided into smaller blocks. The map $(-)|_\lambda$ reflects this division: on the level of standard and irreducible modules, modules that correspond to C's in the same (W_Θ, W_λ)-coset are in the same block, and each block looks like an integral Whittaker category (at least on the level of standard and irreducible modules) modeled by $W_{\lambda,\Theta(u,\lambda)}\backslash W_\lambda$.

We also need to understand how $(-)_\lambda$ behaves under right multiplication by a non-integral simple reflection. This reflects the effect of non-integral intertwining functors which will be defined in §3 and will be used in the algorithm. Roughly speaking, right multiplication by a non-integral simple reflection translates (W_Θ, W_λ)-coset structures to $(W_\Theta, W_{s_\beta\lambda})$-coset structures, while conjugation by the same reflection translates right $W_{\lambda,\Theta(u,\lambda)}$-coset structures in W_λ to $W_{s_\beta\lambda,\Theta(r,s_\beta\lambda)}$-coset structures in $W_{s_\beta\lambda}$.

Lemma 2.4.6. *Let* $u \in A_{\Theta,\lambda}$*,* $\beta \in \Pi - \Pi_\lambda$*. Then* $W_\Theta(us_\beta)$ *is the smallest right* W_Θ*-coset in* $W_\Theta(us_\beta)W_{s_\beta\lambda} = (W_\Theta u W_\lambda)s_\beta$*.*

Proof. By Lemma 2.1.3(e)(f), $us_\beta \in A_{s_\beta\lambda}$. Proposition 2.3.2 says that elements in $A_{s_\beta\lambda}$ are concentrated on the smallest right W_Θ-cosets. So the right coset $W_\Theta(us_\beta)$ containing us_β must be the smallest in the double coset $W_\Theta(us_\beta)W_{s_\beta\lambda}$ containing us_β. This proves the lemma. The final identification simply follows from $s_\beta W_{s_\beta\lambda} s_\beta = W_\lambda$ by 2.1.3(d). □

Rephrasing slightly and using 2.3.2 again, we get

Corollary 2.4.7. *Let* $u \in A_{\Theta,\lambda}$*,* $\beta \in \Pi - \Pi_\lambda$*. If* $r \in A_{\Theta,s_\beta\lambda}$ *denotes the unique representative of* $W_\Theta(us_\beta)W_{s_\beta\lambda}$*, then* $W_\Theta r = W_\Theta us_\beta$*.*

Proposition 2.4.8. *Let* $u \in A_{\Theta,\lambda}$, $\beta \in \Pi - \Pi_\lambda$. *Let* r *be the unique element in* $A_{\Theta,s_\beta\lambda} \cap W_\Theta u s_\beta W_{s_\beta\lambda}$. *Then conjugation by* s_β *is a bijection*

$$s_\beta(-)s_\beta : W_{(s_\beta\lambda),\Theta(r,s_\beta\lambda)}\backslash W_{(s_\beta\lambda)} \xrightarrow{\sim} W_{\lambda,\Theta(u,\lambda)}\backslash W_\lambda$$

that preserves the partial orders on right cosets. Moreover, the following diagram commutes

$$\begin{array}{ccc} W_{(s_\beta\lambda),\Theta(r,s_\beta\lambda)}\backslash W_{(s_\beta\lambda)} & \xrightarrow{s_\beta(-)s_\beta} & W_{\lambda,\Theta(u,\lambda)}\backslash W_\lambda \\ \downarrow{\scriptstyle \mathrm{ind}_{s_\beta\lambda}} & & \downarrow{\scriptstyle \mathrm{ind}_\lambda} \\ W_\Theta\backslash W & \xrightarrow{(-)s_\beta} & W_\Theta\backslash W \end{array} . \tag{2.4.9}$$

In particular, for any C, D *in the image* $\mathrm{ind}_{s_\beta\lambda}\left(W_{(s_\beta\lambda),\Theta(r,s_\beta\lambda)}\backslash W_{(s_\beta\lambda)}\right)$,

$$D \leqslant_{r,s_\beta\lambda} C \iff Ds_\beta \leqslant_{u,\lambda} Cs_\beta.$$

Proof. By the preceding corollary, there exists $w \in W_\Theta$ such that $wr = us_\beta$. Therefore

$$\begin{aligned} s_\beta\Theta(u,\lambda) &= s_\beta(u^{-1}\Sigma_\Theta \cap \Pi_\lambda) && \text{(by definition of } \Theta(u,\lambda)) \\ &= (us_\beta)^{-1}\Sigma_\Theta \cap s_\beta\Pi_\lambda \\ &= (wr)^{-1}\Sigma_\Theta \cap \Pi_{s_\beta\lambda} && \text{(since } wr = us_\beta) \\ &= r^{-1}(w^{-1}\Sigma_\Theta) \cap \Pi_{s_\beta\lambda} \\ &= r^{-1}\Sigma_\Theta \cap \Pi_{s_\beta\lambda} && \text{(since } w \in W_\Theta) \\ &= \Theta(r, s_\beta\lambda) && \text{(by definition of } \Theta(r,s_\beta\lambda)). \end{aligned}$$

Hence conjugation by s_β sends $W_{s_\beta\lambda,\Theta(r,s_\beta\lambda)}$ to $W_{\lambda,\Theta(u,\lambda)}$ and therefore induces a bijection from $W_{s_\beta\lambda,\Theta(r,s_\beta\lambda)}\backslash W_{s_\beta\lambda}$ to $W_{\lambda,\Theta(u,\lambda)}\backslash W_\lambda$. Furthermore, since conjugation by s_β preserves Bruhat orders (Corollary 2.1.4), it also preserves the partial orders on right cosets.

To check that the diagram commutes, take any $D' \in W_{(s_\beta\lambda),\Theta(r,s_\beta\lambda)}\backslash W_{(s_\beta\lambda)}$. Along the top-right path, D' is sent to

$$W_\Theta u \cdot s_\beta D' s_\beta = W_\Theta w r D' s_\beta = W_\Theta r D' s_\beta,$$

which agrees with the image along the bottom-left path. □

2.5 A technical lemma

In the last part of this chapter, we prove a technical lemma that will be used in §4.6 in induction process.

Proposition 2.5.1. *Let* $u \in A_{\Theta,\lambda}$ *and* $C \in W_\Theta \backslash W_\Theta u W_\lambda$. *Suppose* $C \neq W_\Theta u$. *Then there exist* $\alpha \in \Pi_\lambda$, $s \geqslant 0$ *and* $\beta_1, \ldots, \beta_s \in \Pi$ *such that, writing* $z_0 = 1$, $z_i = s_{\beta_1} \cdots s_{\beta_i}$ *and* $z = z_s$, *the following conditions hold:*

(a) *for any* $0 \leqslant i \leqslant s-1$, β_{i+1} *is non-integral to* $z_i^{-1}\lambda$;

(b) $z^{-1}\alpha \in \Pi \cap \Pi_{z^{-1}\lambda}$;

(c) $Cs_\alpha <_{u,\lambda} C$;

(d) *if* $s > 0$, $Cz < C$;

(e) $Cs_\alpha z = Czs_{z^{-1}\alpha} < Cz$.

This proposition is used in showing that the q-polynomials defined geometrically (by taking higher inverse images of irreducible $\mathcal{D}$-modules to Schubert cells) agree with the Whittaker Kazhdan-Lusztig polynomials for the parabolic system $(W_\lambda, \Pi_\lambda, \Theta(u,\lambda))$. This is a proof by induction in the length of C. As mentioned in §1.4, one of the characterizing properties of the Kazhdan-Lusztig basis C_w is a condition on the product $C_w C_s$. An analogous characterization holds for their Whittaker version. If the simple reflection $s \in W_\lambda$ happens to be simple in W, then multiplication by C_s on C_w lifts to the geometric U-functor (push-pull along $X \to X_s$) which has been treated by Romanov. However, if s is not simple in W, no such U-functor exists. The strategy in this situation is to use non-integral intertwining functors to translate everything (this is condition (a) of the proposition) so that s becomes simple in both the integral Weyl group and in W (this is condition (b)). On the W_λ level, these non-integral intertwining functors correspond to applying conjugations $s_{\beta_i}(-)s_{\beta_i}$ by non-integral simple reflections so that $s \in W_\lambda$ is translated to $z^{-1}sz$ which is simple in $W_{z^{-1}\lambda}$. On the W level, they correspond to right multiplication on C by $z = s_{\beta_1} \cdots s_{\beta_s}$. Also, one needs to ensures that the length of C decreases after these non-integral reflections in order to apply the induction hypothesis on C (this is condition (d)). The existence of such a chain of non-integral reflections is guaranteed by the proposition.

Proof. Since $C \neq W_\Theta u$, in particular $C \neq W_\Theta$, there exists a simple reflection s_γ such that $Cs_\gamma < C$.

If there exists $\alpha \in \Pi \cap \Pi_\lambda$ such that $Cs_\alpha < C$, then this α together with $s = 0$ satisfies the requirement: (a) and (d) are void, while (b) and (e) are true by construction. We need to verify (c). Since s_α is simple in (W_λ, Π_λ), we have three mutually exclusive possibilities: $Cs_\alpha) <_{u,\lambda} C$, $Cs_\alpha = C$, or $Cs_\alpha >_{u,\lambda} C$. Since the map ind_λ preserves the partial order, they imply $Cs_\alpha < C$, $Cs_\alpha = C$ and $Cs_\alpha > C$, respectively. By our choice of α, the last two possibilities cannot happen. Hence we must have $Cs_\alpha <_{u,\lambda} C$ and (c) holds.

Suppose such α does not exist. Then any simple reflection that decreases the length of C via right multiplication must be non-integral to λ. Let s_{β_1}, $\beta_1 \in \Pi - \Pi_\lambda$, be one of those. If there exists $\alpha' \in \Pi \cap \Pi_{s_{\beta_1}\lambda}$ with $Cs_{\beta_1}s_{\alpha'} < Cs_{\beta_1}$, we claim that $\alpha := s_{\beta_1}\alpha' \in s_{\beta_1}\Pi_{s_{\beta_1}\lambda} = \Pi_\lambda$, $s = 1$ and β_1 satisfy our requirements. (a) and (d) follows by our choice of s_{β_1}, (e) follows from the conditions on α'. For (b),

$$z^{-1}\alpha = s_{\beta_1}s_{\beta_1}\alpha' = \alpha' \in \Pi \cap \Pi_{s_{\beta_1}\lambda}$$

by definition of z and α'. For (c), arguing in the same way, we only need to rule out $Cs_\alpha \geqslant C$, which would imply $\ell(C) - \ell(Cs_\alpha s_{\beta_1}) \in \{-2, -1, 0, 1\}$. On the other hand,

$$C > Cs_{\beta_1} > Cs_{\beta_1}s_{\alpha'} = Cs_{\beta_1}s_{(s_{\beta_1}\alpha)} = Cs_{\beta_1}s_{\beta_1}s_\alpha s_{\beta_1} = Cs_\alpha s_{\beta_1}.$$

So $\ell(C) - \ell(Cs_\alpha s_{\beta_1}) \geqslant 2$ and (c) holds.

If such α' does not exist, then we can find $\beta_2, \ldots, \beta_s \in \Pi$ such that $Cz_{i+1} < Cz_i$ for all $1 \leqslant i \leqslant s-1$ until we get to a point where there exists $\alpha'' \in \Pi \cap \Pi_{z^{-1}\lambda}$ with $Czs_{\alpha''} < Cz$ (termination of this process is proven in the next paragraph). We claim that $\alpha := z\alpha'' \in z\Pi_{z^{-1}\lambda} = \Pi_\lambda$, s and $\beta_1, \ldots, \beta_s$ satisfy our requirements. The verification is essentially the same as in the previous case. (a), (b), (d) and (e) are satisfied by our choice of β_i's and α''. For (c), we have an inequality

$$Cz > Czs_{\alpha''} = Czs_{z^{-1}\alpha} = Czz^{-1}s_\alpha z = Cs_\alpha z \tag{2.5.2}$$

where $\ell(w^{Cz}) = \ell(w^C z) = \ell(w^C) - s$. Also $w^{Czs_{\alpha''}} = w^C zs_{\alpha''} = w^C s_\alpha z$. Hence

$$\begin{aligned}
\ell(w^C s_\alpha) &= \ell(w^C s_\alpha zz^{-1}) \\
&= \ell(w^{Cs_\alpha z}z^{-1}) \\
&\leqslant \ell(w^{Cs_\alpha z}) + s \\
&= \ell(w^{Cz}) - 1 + s \\
&= \ell(w^C) - 1 < \ell(w^C).
\end{aligned}$$

This rules out $Cs_\alpha \geqslant C$ and (c) is thus verified.

Lastly, let us show that this process of finding α'' must terminate no later than when we get to $\ell(w^{Cz}) = \ell(w_\Theta) + 1$. That is, we show that when $\ell(w^{Cz}) = \ell(w_\Theta) + 1$, such an α'' must exist. The condition $\ell(w^{Cz}) = \ell(w_\Theta) + 1$ implies $Cz = W_\Theta s_\gamma > W_\Theta$ for some simple reflection s_γ. If $\gamma \in \Pi - \Pi_{z^{-1}\lambda}$, then $s_\gamma \in A_{z^{-1}\lambda}$. Also, since $W_\Theta s_\gamma > W_\Theta$, any element of $W_\Theta s_\gamma$ must have length $\geqslant 1$. Hence s_γ is the shortest element of $W_\Theta s_\gamma$, i.e. $s_\gamma \in w_\Theta{}^\Theta W$. Therefore $s_\gamma \in A_{z^{-1}\lambda} \cap (w_\Theta{}^\Theta W) = A_{\Theta, z^{-1}\lambda}$. Since $C = W_\Theta s_\gamma z^{-1}$, by (repeatedly applying) Lemma 2.4.6, we see that C is the smallest right W_Θ-coset in the (W_Θ, W_λ)-coset containing it, that is, $C = W_\Theta u$. This contradicts our assumption on C. Therefore $\gamma \in \Pi \cap \Pi_\lambda$, and $\alpha'' = \gamma$ satisfies our requirement for α''. Thus the process terminates. $\square$

Chapter 3

Intertwining functors and the $\mathbb{U}$-functor

In this section, we study intertwining functors and the $\mathbb{U}$-functor. §3.1 defines these functors and presents a proof of the structure of $\mathbb{U}$-functor on transversal irreducible modules. The proof is streamlined from the one in [Mil] which was partially reproduced by Romanov in [Rom21]. §3.2 studies intertwining functors for non-integral reflections and show that they translate the Kazhdan-Lusztig polynomials.

Readers can review §1.3 for the basic geometric setup and related notations. In the rest of the paper, we will use facts about $\mathcal{D}$-modules without citing references, including the distinguished triangle for immersions of a smooth closed subvariety and its complement (also known as the distinguished triangle for local cohomology), the base change theorem for $\mathcal{D}$-modules, and Kashiwara's equivalence of categories for closed immersions. These facts are contained in [BGK+87], IV.8.3, 8.4 and 7.11, respectively.

3.1 Definitions of the functors and their action on irreducible modules

Write θ for a Weyl group orbit in $\mathfrak{h}^*$, and let $\lambda \in \theta$. Write $D^b(\mathcal{U}_\theta) = D^b(\mathrm{Mod}(\mathcal{U}_\theta))$. For a twisted sheaf of differential operators $\mathcal{D}$ on a space, write $D^b(\mathcal{D}) = D^b(\mathrm{Mod}_{qc}(\mathcal{D}))$ for the bounded derived category of quasi-coherent $\mathcal{D}$-modules.[1] Recall that the localization theorem of Beilinson and Bernstein - an equivalence of categories

$$\mathcal{D}_\lambda \underset{\mathcal{U}_\theta}{\otimes} - : \mathrm{Mod}(\mathcal{U}_\theta) \cong \mathrm{Mod}_{qc}(\mathcal{D}_\lambda) : \Gamma(X, -)$$

for antidominant regular λ. For $w\lambda$ ($w \in W$, still for λ antidominant regular) at another Weyl chamber, taking global sections is no longer exact, but its amplitude is controlled by w. Namely, for any quasi-coherent $\mathcal{D}_{w\lambda}$-module $\mathcal{V}$, $H^i(X, \mathcal{V})$ can be nonzero only in degrees between 0 and $\ell(w)$.

[1] When using derived functors, one needs to be careful of which derived category to work in. For example, one may choose to instead work with the full subcategory of the derived category of all $\mathcal{D}$-modules with quasi-coherent cohomologies $D^b_{qc}(\mathrm{Mod}(\mathcal{D}))$, as is done in [HTT08]. These issues have been carefully cleaned up in [Mil, Chapter 3 §1].

The above equivalence of categories of abelian categories now becomes an equivalence of derived categories

$$D^b(\mathcal{U}_\theta) \cong D^b(\mathcal{D}_{w\lambda}).$$

Therefore, we have an equivalence

$$D^b(\mathcal{D}_\lambda) \cong D^b(\mathcal{U}_\theta) \cong D^b(\mathcal{D}_{w\lambda}).$$

The intertwining functors LI_w are geometrically constructed functors that realize the above equivalence without going through $\mathcal{U}_\theta$.

In more details, for any $w \in W$, let Z_w denote the G-orbit in $X \times X$ labeled by w. This is the subset of $X \times X$ consisting of pairs (x, y) such that the Borel subalgebras $\mathfrak{b}_x$ and $\mathfrak{b}_y$ corresponding to x and y are in relative position w. Here $\mathfrak{b}_x$ and $\mathfrak{b}_y$ are in *relative position* w if, for any common Cartan subalgebra $\mathfrak{c}$, the sets of positive roots defined by $\mathfrak{b}_x$ and $\mathfrak{b}_y$ differ by w. If w is fixed, we write

$$X \xleftarrow{p_1} Z_w \xrightarrow{p_2} X$$

for the two projections. For an integral weight $\mu \in \mathfrak{h}^*$, write $\mathcal{O}_X(\mu)$ for the G-equivariant line bundle on X where the $\mathfrak{b}_x$-action on the geometric fiber at $x \in X$ is given by μ. Tensoring with $\mathcal{O}_X(\mu)$ is an equivalence of categories

$$- \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_X(\mu) : \mathrm{Mod}_{qc}(\mathcal{D}_\lambda) \cong \mathrm{Mod}_{qc}(\mathcal{D}_{\lambda+\mu})$$

which we simply denote by $\mathcal{V} \otimes_{\mathcal{O}_X} \mathcal{O}_X(\mu) =: \mathcal{V}(\mu)$. Twisting by line bundles shares the usual properties with respect to direct and inverse images (for example, the projection formula holds). It will not play a substantial role for us other than book-keeping purposes.

Definition 3.1.1. For $w \in W$ and $\lambda \in \mathfrak{h}^*$, the **intertwining functor** LI_w is defined to be

$$LI_w : D^b(\mathcal{D}_\lambda) \to D^b(\mathcal{D}_{w\lambda}),$$
$$\mathcal{F}^\bullet \mapsto p_{1+}\big(\mathcal{E} \underset{\mathcal{O}_{Z_w}}{\otimes} p_2^+ \mathcal{F}^\bullet\big)$$

where $\mathcal{E}$ is the unique line bundle on Z_w that ensures we land in the correct category. Explicitly, $\mathcal{E} = p_1^* \mathcal{O}_X(\rho - w\rho)$.

Write I_w for $H^0 LI_w$. It is shown in [Mil, L.3] that LI_w is the left derived functor of I_w.

Theorem 3.1.2 ([Mil, Ch.3 §3])**.** *Let $w \in W$ be arbitrary. Then*

(1) The left cohomological dimension of LI_w *is* $\leqslant \ell(w)$*;*

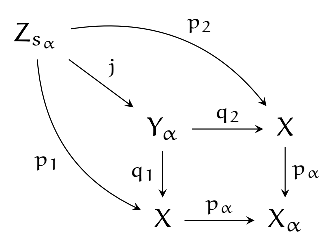
(2) LI_w *is an equivalence of categories;*

(3) If λ *is antidominant, the functors* $\Gamma(X, -)$ *and* $R\Gamma(X, -) \circ \mathrm{LI}_w$ *from* $D^b(\mathcal{D}_\lambda)$ *to* $D^b(\mathrm{Mod}(\mathcal{U}_\theta))$ *are isomorphic;*

(4) If $\beta \in \Pi - \Pi_\lambda$, I_{s_β} *is an equivalence of categories* $\mathrm{Mod}_{qc}(\mathcal{D}_\lambda) \cong \mathrm{Mod}_{qc}(\mathcal{D}_{s_\beta \lambda})$ *whose quasi-inverse is also given by* I_{s_β}.

We will mainly look at intertwining functors for a simple reflection $w = s_\alpha$. The behavior of LI_{s_α} differs greatly depending on the integrality of α. We study the integral case in this section. The non-integral case will be treated in the next section.

For the rest of this section, let $\alpha \in \Pi \cap \Pi_\lambda$, i.e. a simple root integral to λ. LI_{s_α} is naturally related to two other functors through the following diagram. The closure of Z_{s_α} in $X \times X$ is the union $Y_\alpha := \Delta_X \cup Z_{s_\alpha}$ which fits into the following commutative diagram



(3.1.3)

where X_α is the partial flag variety of type s_α, and the square is Cartesian. Using the variety Y_α, we define

Definition 3.1.4.

$$\begin{aligned} U : \mathrm{Mod}_{qc}(\mathcal{D}_\lambda) &\to D^b(\mathcal{D}_{s_\alpha \lambda}) \\ \mathcal{V} &\mapsto q_{1+}\big(\mathcal{E} \underset{\mathcal{O}_{Y_\alpha}}{\otimes} q_2^+ \mathcal{V}\big), \end{aligned}$$

where $\mathcal{E}$ is the unique line bundle that ensures we land in the correct category. Explicitly,

$$\mathcal{E} = q_1^* \mathcal{O}_X\big((\alpha^\vee(\lambda) + 1)\rho - \alpha^\vee(\lambda)\alpha\big) \underset{\mathcal{O}_{Y_\alpha}}{\otimes} q_2^* \mathcal{O}_X\big((\alpha^\vee(\lambda) + 1)\rho\big).$$

Since q_2 is flat and q_1 has relative dimension 1, $H^j U \mathcal{V}$ can be nonzero only for $-1 \leqslant j \leqslant 1$.

We want to define a similar functor going through X_α instead. This requires the existence of a twisted sheaf of differential operators $\mathcal{D}_{X_\alpha, \lambda}$ on X_α whose pullback to X is $\mathcal{D}_\lambda$. Such existence is equivalent to $\alpha^\vee(\lambda) = -1$. Since α is assumed to be integral to λ, we can find an integral weight μ_α such that $\alpha^\vee(\lambda + \mu_\alpha) = -1$.

Definition 3.1.5. Let $\mu_\alpha \in \mathfrak{h}^*$ be an integral weight such that $\alpha^\vee(\lambda + \mu_\alpha) = -1$. Write $\mathcal{D}_{X_\alpha, \lambda+\mu_\alpha}$ for the twisted sheaf of differential operators on X_α determined by $\lambda + \mu_\alpha$. Define U_α to be the composition

$$\mathrm{Mod}_{qc}(\mathcal{D}_\lambda) \xrightarrow{-\otimes \mathcal{O}_X(\mu_\alpha)} \mathrm{Mod}_{qc}(\mathcal{D}_{\lambda+\mu_\alpha}) \xrightarrow{p_{\alpha+}} D^b_{qc}(\mathcal{D}_{X_\alpha,\lambda+\mu_\alpha})$$
$$\xrightarrow{p_\alpha^+} D^b(\mathcal{D}_{\lambda+\mu_\alpha}) \xrightarrow{-\otimes \mathcal{O}_X(-\mu_\alpha)} D^b(\mathcal{D}_\lambda)$$

that is,

$$U_\alpha : \mathrm{Mod}_{qc}(\mathcal{D}_\lambda) \to D^b(\mathcal{D}_\lambda),$$
$$\mathcal{V} \mapsto \left(p_\alpha^+ p_{\alpha+} \mathcal{V}(\mu_\alpha)\right)(-\mu_\alpha).$$

This does not depend on the choice of μ_α.

Since p_α is flat and has relative dimension 1, $H^j U_\alpha$ can be nonzero only if $-1 \leqslant j \leqslant 1$. By base-changing using the Cartesian square in (3.1.3), we see that U_α is a twist of U.

Lemma 3.1.6. *For any $\mathcal{D}_\lambda$-module $\mathcal{V}$, and any $\alpha \in \Pi_\lambda \cap \Pi$, $(U_\alpha \mathcal{V})(-\alpha^\vee(\lambda)\alpha) = U\mathcal{V}$.*

Remark 3.1.7. $H^0 U_\alpha$ is the geometric version of Vogan's U-functor defined in [Vog79, Definition 3.8], but we will not need this fact.

The main result of this section is the following.

Theorem 3.1.8 ([Mil, Ch.5 Lemma 2.7], [Rom21, Lemma 17])**.** *Let $C \in W_\Theta \backslash W$ and $\alpha \in \Pi_\lambda \cap \Pi$ such that $Cs_\alpha < C$. Then*

(a) For all p, $H^p U_\alpha \mathcal{L}(w^C s_\alpha, \lambda, \eta)$ is a direct sum of $\mathcal{L}(w^D, \lambda, \eta)$'s for some $D \leqslant C$,

(b) for all $p \neq 0$, $H^p U_\alpha \mathcal{L}(w^C s_\alpha, \lambda, \eta) = 0$, and

(c) $\mathcal{L}(w^C, \lambda, \eta)$ is a direct summand of $H^0 U_\alpha \mathcal{L}(w^C s_\alpha, \lambda, \eta)$.

In particular, there exist integers c_D's for each $D \leqslant C$, depending on C and α, so that

$$U_\alpha \mathcal{L}(w^C s_\alpha, \lambda, \eta) = \bigoplus_{D \leqslant C} \mathcal{L}(w^D, \lambda, \eta)^{\oplus c_D}.$$

Remark 3.1.9. In Chapter 4 we will be able to obtain a more precise vanishing. Namely, irreducibles $\mathcal{L}(w^D, \lambda, \eta)$ that are not in the same block as $\mathcal{L}(w^C, \lambda, \eta)$ will not appear in $U_\alpha \mathcal{L}(w^C s_\alpha, \lambda, \eta)$. But this is a consequence of the main algorithm and does not follow from looking at U_α itself.

Because of the condition $Cs_\alpha < C$ (we say that s_α is *transversal* to $C(w^C s_\alpha)$), the push-pull operation enlarges the support of $\mathcal{L}(w^C s_\alpha, \lambda, \eta)$ by one dimension. So part (c) is natural (modulo the part that $\mathcal{L}(w^C, \lambda, \eta)$ appears only in H^0). Part (a) follows easily from the decomposition theorem. Part (b) is more subtle.

To ease notation, we write $w = w^C$, and we omit writing η from now on in the proofs. We will also stop writing the line bundle twists so long as the categories we are working with are clear. Their appearances in the previous definitions is entirely for book-keeping purpose, and it is easy to recover them in the proofs.

Proof of 3.1.8(a). Part (a) follows from the Decomposition Theorem for holonomic $\mathcal{D}$-modules, proven by Mochizuki [Moc11]. In more detail, Decomposition Theorem says that direct image of an irreducible holonomic $\mathcal{D}$-module along a proper morphism is a direct sum of irreducible modules in various degrees. Applied to the proper morphism $p_\alpha : X \to X_\alpha$ and to the irreducible module $\mathcal{L}(ws_\alpha, \lambda)$, we see that $H^p p_{\alpha+}\mathcal{L}(ws_\alpha, \lambda)$ for any $p \in \mathbb{Z}$ is a direct sum of irreducible $\mathcal{D}$-modules $\mathcal{J}$'s on X_α. So $H^p U_\alpha \mathcal{L}(ws_\alpha, \lambda)$ is a direct sum of (line bundle twists of) $p_\alpha^+\mathcal{J}$'s (here we used the fact that p_α^+ is exact and commutes with taking H^p). We need to show that each $p_\alpha^+\mathcal{J}$ is irreducible.

Since $\mathcal{J}$ is irreducible, it has irreducible support. Since p_α is a locally trivial fibration, $p_\alpha^+\mathcal{J}$ has irreducible support, and locally it is irreducible or zero. Suppose $p_\alpha^+\mathcal{J}$ has a proper submodule $\mathcal{W}$, so $p_\alpha^+\mathcal{J}$ fits into a short exact sequence

$$0 \longrightarrow \mathcal{W} \longrightarrow p_\alpha^+\mathcal{J} \longrightarrow (p_\alpha^+\mathcal{J})/\mathcal{W} \longrightarrow 0.$$

Then $\operatorname{Supp} p_\alpha^+\mathcal{J} = \operatorname{Supp}\mathcal{W} \cup \operatorname{Supp}(p_\alpha^+\mathcal{J})/\mathcal{W}$. We claim that $\operatorname{Supp}\mathcal{W}$ and $\operatorname{Supp}(p_\alpha^+\mathcal{J})/\mathcal{W}$ must be disjoint. Assume otherwise, then we can take an open set $U \subset X$ that contains a point of $\operatorname{Supp}\mathcal{W} \cap \operatorname{Supp}(p_\alpha^+\mathcal{J})/\mathcal{W}$ and so that $(p_\alpha^+\mathcal{J})|_U$ is irreducible. But then we would have a short exact sequence

$$0 \longrightarrow \mathcal{W}|_U \longrightarrow (p_\alpha^+\mathcal{J})|_U \longrightarrow \big((p_\alpha^+\mathcal{J})/\mathcal{W}\big)|_U \longrightarrow 0$$

with all terms nonzero irreducible, which is impossible. This proves the claim. On the other hand, if $\operatorname{Supp}\mathcal{W}$ and $\operatorname{Supp}(p_\alpha^+\mathcal{J})/\mathcal{W}$ are disjoint, $\operatorname{Supp} p_\alpha^+\mathcal{J}$ is now reducible, again a contradiction. Therefore $p_\alpha^+\mathcal{J}$ must be irreducible. As a result, $H^p U_\alpha \mathcal{L}(ws_\alpha, \lambda)$ is a direct sum of irreducible modules. Since these modules are all η-twisted weakly N-equivariant, they all take the form $\mathcal{L}(w^D, \lambda)$.

It remains to show that modules that appear must satisfy $D \leqslant C$. This is again a support argument. In view of the definition of U_α, twisting by a line bundle does not change the support, while $\operatorname{Supp} H^p p_\alpha^+ p_{\alpha+}\mathcal{V}$ is contained in $p_\alpha^{-1}(p_\alpha(\operatorname{Supp}\mathcal{V}))$ for any module $\mathcal{V}$. In our case we are looking at $\operatorname{Supp}\mathcal{V} = \operatorname{Supp}\mathcal{L}(ws_\alpha, \lambda) = \overline{C(w^C s_\alpha)}$, the closure of the Schubert cell labeled by $w^C s_\alpha$.

By our assumption on C, $ws_\alpha < (ws_\alpha)s_\alpha = w$. So p_α maps $C(ws_\alpha)$ isomorphically onto its image, and $p_\alpha^{-1}(p_\alpha(C(ws_\alpha)))$ equals the union $C(ws_\alpha) \cup C(w)$ in which $C(w)$ is open. Therefore the support of $H^p U_\alpha \mathcal{L}(ws_\alpha, \lambda)$ is contained in the closure of $C(w)$. This forces any direct summand to be supported in $\overline{C(w)}$ and hence must have $D \leqslant C$. This proves (a). □

From this proof, we also see that any direct summand $\mathcal{L}$ of $H^p U_\alpha \mathcal{L}(ws_\alpha, \lambda)$ is of the form $p_\alpha^+ \mathcal{J}$. So its support will be of the form $p_\alpha^{-1}(\operatorname{Supp} \mathcal{J})$, which saturates any fiber of p_α meeting it. We record this as a lemma for later use.

Lemma 3.1.10. *Let $p \in \mathbb{Z}$, let F be a fiber of p_α, and let $\mathcal{L}$ be a direct summand of $H^p U_\alpha \mathcal{L}(w^C s_\alpha, \lambda, \eta)$. Then either*

- $\operatorname{Supp} \mathcal{L} \cap F = \varnothing$, *or*
- $\operatorname{Supp} \mathcal{L} \cap F = F$.

Part (b) is harder because it requires one to actually compute cohomologies of $p_{\alpha+}\mathcal{L}(ws_\alpha, \lambda)$, which is in general a difficult problem. We will get around this difficulty by first relating U_α to the intertwining functor LI_{s_α} and exploit the fact that the latter plays well with localization (namely Theorem 3.1.2).

We first examine the relation between U_α and LI_{s_α}. Recall the diagram

$$\begin{array}{ccccc} Z_{s_\alpha} & & \xrightarrow{\quad p_2 \quad} & & X \\ & \searrow^{j} & & \nearrow & \\ & & Y_\alpha & \xrightarrow{q_2} & X \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle q_1} & & \downarrow{\scriptstyle p_\alpha} \\ & \searrow & X & \xrightarrow{p_\alpha} & X_\alpha \end{array} \quad . \tag{3.1.3}$$

As remarked in 3.1.6, U_α and U differ only by a twist. Hence the vanishing of $H^p U_\alpha$ on $\mathcal{L}(ws_\alpha, \lambda)$ is equivalent to that of $H^p U \mathcal{L}(ws_\alpha, \lambda)$. On the other hand, U can be related to the intertwining functor I_{s_α} in the following way. Recall that the variety $Y_\alpha = X \times_{X_\alpha} X$ used to define the functor U is the union $\Delta_X \cup Z_{s_\alpha}$. So we have immersions

$$\Delta_X \xrightarrow{i} Y_\alpha \xleftarrow{j} Z_{s_\alpha}.$$

The distinguished triangle with respect to these immersions reads

$$\begin{array}{ccc} i_+ i^! & \longrightarrow & \mathrm{Id} \\ & {}_{[1]}\nwarrow \quad \swarrow & \\ & j_+ j^! & \end{array} \quad .$$

For a $\mathcal{D}_\lambda$-module $\mathcal{V}$ on X (which will be specified to $\mathcal{V} = \mathcal{L}(ws_\alpha, \lambda)$ later), we can apply this triangle to $q_2^+\mathcal{V}$:

$$\begin{array}{ccc} i_+ i^! q_2^+ \mathcal{V} & \longrightarrow & q_2^+\mathcal{V} \\ {\scriptstyle [1]} \nwarrow & & \swarrow \\ & j_+ j^! q_2^+ \mathcal{V} & \end{array} .$$

Note that $q_2 \circ i : \Delta_X \to X$ is an isomorphism, and $q_2 \circ j = p_2$. This triangle then simplifies to

$$\begin{array}{ccc} i_+ \mathcal{V}[-1] & \longrightarrow & q_2^+\mathcal{V} \\ {\scriptstyle [1]} \nwarrow & & \swarrow \\ & j_+ p_2^+ \mathcal{V} & \end{array} .$$

Now apply q_{1+}, we get

$$\begin{array}{ccc} q_{1+} i_+ \mathcal{V}[-1] & \longrightarrow & q_{1+} q_2^+\mathcal{V} \\ {\scriptstyle [1]} \nwarrow & & \swarrow \\ & q_{1+} j_+ p_2^+ \mathcal{V} & \end{array}$$

in $D^b(\mathcal{D}_{s_\alpha\lambda})$, which simplifies to

$$\begin{array}{ccc} \mathcal{V}(-\alpha^\vee(\lambda)\alpha)[-1] & \longrightarrow & U\mathcal{V} \\ {\scriptstyle [1]} \nwarrow & & \swarrow \\ & LI_{s_\alpha} \mathcal{V} & \end{array}$$

noting that $q_1 \circ i : \Delta_X \to X$ is an isomorphism and $q_1 \circ j = p_1$ (the twist at the top left corner comes from remembering all the line bundle twists we omitted). The long exact sequence of sheaf cohomologies then splits into two sequences:

Proposition 3.1.11. *Let $\mathcal{V} \in \mathrm{Mod}_{qc}(\mathcal{D}_\lambda)$ and $\alpha \in \Pi \cap \Pi_\lambda$. Then we have the following to exact sequences*

$$0 \longrightarrow H^{-1}U\mathcal{V} \longrightarrow L^{-1}I_{s_\alpha}\mathcal{V} \longrightarrow 0, \tag{3.1.12}$$

$$0 \longrightarrow H^0 U\mathcal{V} \longrightarrow I_{s_\alpha}\mathcal{V} \longrightarrow \mathcal{V}(-\alpha^\vee(\lambda)\alpha) \longrightarrow H^1 U\mathcal{V} \longrightarrow 0. \tag{3.1.13}$$

The vanishing of $H^{\pm 1}U\mathcal{V}$ will result from information on $LI_{s_\alpha}\mathcal{V}$, which we compute now for $\mathcal{V} = \mathcal{L}(ws_\alpha, \lambda)$.

Proposition 3.1.14. *Let $\mathcal{V}$ be an irreducible $\mathcal{D}_\lambda$-module. Then exactly one of the following happens: either*

- $I_{s_\alpha}\mathcal{V} = 0$, *or*
- $L^{-1}I_{s_\alpha}\mathcal{V} = 0$.

Proof. Since intertwining functors plays well with line bundle twists, without loss of generalities we can assume λ is antidominant regular, so that localization theorems can be used. Write $V = \Gamma(X, \mathcal{V})$.

From Theorem 3.1.2 we know that $R\Gamma(X, LI_{s_\alpha}\mathcal{V}) = V$. On the other hand, we have a spectral sequence

$$E_2^{p,q} = H^q(X, L^q I_{s_\alpha}\mathcal{V}) \implies H^{p+q}R\Gamma(X, LI_{s_\alpha}\mathcal{V}) = H^{p+q}V.$$

By Theorem 3.1.2, $LI_{s_\alpha}\mathcal{V}$ is concentrated in degrees -1 and 0, whose cohomologies are $\mathcal{D}_{s_\alpha\lambda}$-modules. Also, cohomologies of $\mathcal{D}_{s_\alpha\lambda}$-modules vanish outside degree 0 and 1. So the E_2-page is concentrated in degrees $-1 \leqslant p \leqslant 0$ and $0 \leqslant q \leqslant 1$, and so $E_2 = E_\infty$. The right hand side is simply L for $p+q=0$ and 0 otherwise. Hence the spectral sequence tells us

$$\Gamma(X, L^{-1}I_{s_\alpha}\mathcal{V}) = H^1(X, I_{s_\alpha}\mathcal{V}) = 0,$$

and by irreducibility of V, either

- $H^1(X, L^{-1}I_{s_\alpha}\mathcal{V}) = 0$, $\Gamma(X, I_{s_\alpha}\mathcal{V}) = V$, or
- $H^1(X, L^{-1}I_{s_\alpha}\mathcal{V}) = V$, $\Gamma(X, I_{s_\alpha}\mathcal{V}) = 0$.

The first case implies $R\Gamma(X, L^{-1}I_{s_\alpha}\mathcal{V}) = 0$. Since $R\Gamma(X, -)$ is an equivalence of categories between $D^b(\mathcal{D}_{s_\alpha\lambda})$ and $D^b(\mathcal{U}_\theta)$, this implies $L^{-1}I_{s_\alpha}\mathcal{V} = 0$. The second case implies $I_{s_\alpha}\mathcal{V} = 0$ by the same argument. These two cases cannot happen at the same time because LI_{s_α} is an equivalence of categories. ☐

We now show that the second case happens in our case (where $\mathcal{V} = \mathcal{L}(ws_\alpha, \lambda)$ and $ws_\alpha < w$). Write O for the N-orbit $p_\alpha(C(ws_\alpha)) = p_\alpha(C(w))$ in X_α. Its preimage $X_O := p_\alpha^{-1}(O)$ is the union $C(ws_\alpha) \cup C(w)$. Write $s : X_O \to X$ for the inclusion map.

Lemma 3.1.15. *$s^! LI_{s_\alpha}\mathcal{L}(w^C s_\alpha, \lambda, \eta)$ is nonzero and is concentrated in degree 0.*

Proof. Write

$$Z_{s_\alpha,O} := \{(x, x') \in X_O \times X_O \mid \mathfrak{b}_x \text{ and } \mathfrak{b}_{x'} \text{ are in relative position } s_\alpha\}.$$

This naturally sits inside the preimage of X_O in Z_{s_α}:

$$p_2^{-1}(X_O) = \{(x, x') \in X \times X_O \mid \mathfrak{b}_x \text{ and } \mathfrak{b}_{x'} \text{ are in relative position } s_\alpha\}.$$

However, if x and x' are in relative position s_α, then they have the same image in X_α. Hence $x' \in X_O$ implies $p_\alpha(x) = p_\alpha(x') \in O$, and this forces $x \in X_O$. As a result

$$p_2^{-1}(X_O) = Z_{s_\alpha,O}.$$

Hence, we have the following diagram

$$\begin{array}{ccccc} X_O & \xleftarrow{\pi_1} & Z_{s_\alpha,O} & \xrightarrow{\pi_2} & X_O \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle \tilde{s}} & & \downarrow{\scriptstyle s} \\ X & \xleftarrow{p_1} & Z_{s_\alpha} & \xrightarrow{p_2} & X \end{array}$$

where both squares are Cartesian. Moreover, the preimage of $C(ws_\alpha) \subset X_O$ under π_2 is isomorphic to $C(w)$ through the first projection. So the above diagram can be extended to

$$\begin{array}{ccccc} C(w) & = & C(w) & \xrightarrow{\varpi_2} & C(ws_\alpha) \\ {\scriptstyle \iota_w}\downarrow & & \downarrow{\scriptstyle \tilde{\iota}_w} & & \downarrow{\scriptstyle \iota_{ws_\alpha}} \\ X_O & \xleftarrow{\pi_1} & Z_{s_\alpha,O} & \xrightarrow{\pi_2} & X_O \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle \tilde{s}} & & \downarrow{\scriptstyle s} \\ X & \xleftarrow{p_1} & Z_{s_\alpha} & \xrightarrow{p_2} & X \end{array} \tag{3.1.16}$$

where the top-right square is also Cartesian. We use this diagram and base change to compute $s^! LI_{s_\alpha} \mathcal{L}(ws_\alpha, \lambda)$.

$$\begin{aligned} & s^! LI_{s_\alpha} \mathcal{L}(ws_\alpha, \lambda) \\ &= s^! p_{1+} p_2^+ \mathcal{L}(ws_\alpha, \lambda) && \text{(by definition of } LI_{s_\alpha}\text{, omitting twists)} \\ &= \pi_{1+} \tilde{s}^! p_2^+ \mathcal{L}(ws_\alpha, \lambda) && \text{(base changing)} \\ &= \pi_{1+} \tilde{s}^! p_2^! \mathcal{L}(ws_\alpha, \lambda)[d_X - d_{Z_{s_\alpha}}] && \text{(since } p_2^+ = p_2^![d_X - d_{Z_{s_\alpha}}]) \\ &= \pi_{1+} \pi_2^! s^! \mathcal{L}(ws_\alpha, \lambda)[d_X - d_{Z_{s_\alpha}}] && (p_2 \circ \tilde{s} = s \circ \pi_2). \end{aligned} \tag{3.1.17}$$

Here d_X denotes the dimension of X; similarly for $d_{Z_{s_\alpha}}$.

Now we notice that $s^! \mathcal{L}(ws_\alpha, \lambda)$ is supported on the closed subvariety $C(ws_\alpha)$ in X_O. By Kashiwara's equivalence for closed immersions,

$$\begin{aligned} s^! \mathcal{L}(ws_\alpha, \lambda) &= \iota_{ws_\alpha +} \iota_{ws_\alpha}^! s^! \mathcal{L}(ws_\alpha, \lambda) \\ &= \iota_{ws_\alpha +} i_{ws_\alpha}^! \mathcal{L}(ws_\alpha, \lambda), \end{aligned}$$

where the last equality follows from $\iota_{ws_\alpha} \circ s = i_{ws_\alpha}$, the inclusion of $C(ws_\alpha)$ in X.

Lemma 3.1.18. *For any* C, λ, *and* η,

$$i_{w^C}^! \mathcal{L}(w^C, \lambda, \eta) = \mathcal{O}_{C(w^C)}^\eta.$$

Proof of 3.1.18. Since $\mathcal{I}(w^C, \lambda, \eta)$ is a direct image, it contains no section supported in $\partial C(w^C)$ except 0. The same holds for $\mathcal{L}(w^C, \lambda, \eta)$ being a submodule of $\mathcal{I}(w^C, \lambda, \eta)$. Hence

$\mathcal{L}(w^C, \lambda, \eta)|_{X-\partial C(w^C)}$ is a nonzero submodule of $\mathcal{I}(w^C, \lambda, \eta)|_{X-\partial C(w^C)}$. But $\mathcal{I}(w^C, \lambda, \eta)|_{X-\partial C(w^C)}$ is irreducible by Kashiwara's equivalence of categories for the closed immersion $C(w^C) \hookrightarrow (X - \partial C(w^C))$, so $\mathcal{L}(w^C, \lambda, \eta)|_{X-\partial C(w^C)} = \mathcal{I}(w^C, \lambda, \eta)|_{X-\partial C(w^C)}$, and their further pullback to $C(w^C)$ is $\mathcal{O}^{\eta}_{C(w^C)}$. □

Applied to Cs_α, we see that $i^!_{ws_\alpha}\mathcal{L}(ws_\alpha, \lambda) = \mathcal{O}^{\eta}_{C(ws_\alpha)}$. Hence $s^!\mathcal{L}(ws_\alpha, \lambda) = \iota_{ws_\alpha +}\mathcal{O}^{\eta}_{C(ws_\alpha)}$. Therefore, continuing the calculation in (3.1.17),

$$\begin{aligned}
&s^! L I_{s_\alpha}\mathcal{L}(ws_\alpha, \lambda) \\
&= \pi_{1+}\pi_2^!\iota_{ws_\alpha+}\mathcal{O}^{\eta}_{C(ws_\alpha)}[d_X - d_{Z_{s_\alpha}}] \\
&= \pi_{1+}\tilde{\iota}_{w+}\varpi_2^!\mathcal{O}^{\eta}_{C(ws_\alpha)}[d_X - d_{Z_{s_\alpha}}] && \text{(base changing)} \\
&= \pi_{1+}\tilde{\iota}_{w+}\varpi_2^+\mathcal{O}^{\eta}_{C(ws_\alpha)}[d_X - d_{Z_{s_\alpha}} + d_{C(w)} - d_{C(ws_\alpha)}] && (\text{since } \varpi_2^! = \varpi_2^+[d_{C(w)} - d_{C(ws_\alpha)}]) \\
&= \iota_{w+}\mathcal{O}^{\eta}_{C(w)}[d_X - d_{Z_{s_\alpha}} + d_{C(w)} - d_{C(ws_\alpha)}] && (\pi_1 \circ \tilde{\iota}_w = \iota_w) \\
&= \iota_{w+}\mathcal{O}^{\eta}_{C(w)}, && (3.1.19)
\end{aligned}$$

where the last equality is because

$$d_X - d_{Z_{s_\alpha}} + d_{C(w)} - d_{C(ws_\alpha)} = d_X - (d_X + 1) + \ell(w) - (\ell(w) - 1) = 0.$$

As a result, $s^! L I_{s_\alpha}\mathcal{L}(ws_\alpha, \lambda)$ is concentrated in degree 0. □

What does this tell us about vanishing of $L^{-1}I_{s_\alpha}\mathcal{L}(ws_\alpha, \lambda)$? Suppose otherwise, then by Proposition 3.1.14 $LI_{s_\alpha}\mathcal{L}(ws_\alpha, \lambda) = L^{-1}I_{s_\alpha}\mathcal{L}(ws_\alpha, \lambda)[1]$. On the other hand, the inclusion $s : X_O \to X$ decomposes as

$$X_O \xrightarrow{s_{cl}} X - \partial X_O \xrightarrow{s_{op}} X$$

where $\partial X_O = \overline{X_O} - X_O$ is the boundary of X_O, s_{op} is open, and s_{cl} is closed. Hence $s^!_{op}$ has zero amplitude. By Kashiwara's equivalence, $s^!_{cl}$ has zero amplitude on complexes whose cohomologies are supported in X_O. Our $L^{-1}I_{s_\alpha}\mathcal{L}(ws_\alpha, \lambda)$ is supported in the closure of $p_1(p_2^{-1}(C(ws_\alpha)))$, which equals $\overline{C(w)} = \overline{X_O}$. So cohomologies of $s^!_{op}L^{-1}I_{s_\alpha}\mathcal{L}(ws_\alpha, \lambda)$ are supported in X_O. As a result, $s^! = s^!_{cl} \circ s^!_{op}$ has zero amplitude on $L^{-1}I_{s_\alpha}\mathcal{L}(ws_\alpha)$. Therefore $s^! L^{-1}I_{s_\alpha}\mathcal{L}(ws_\alpha, \lambda)[1]$ is either 0 or concentrated in degree -1. This is a contradiction because by previous calculation

$$s^! L^{-1}I_{s_\alpha}\mathcal{L}(ws_\alpha, \lambda)[1] = s^! L I_{s_\alpha}\mathcal{L}(ws_\alpha, \lambda) = \iota_{w+}\mathcal{O}^{\eta}_{C(w)}$$

is nonzero and concentrated in degree 0. Thus $L^{-1}I_{s_\alpha}\mathcal{L}(ws_\alpha, \lambda)$ must vanish.

Proposition 3.1.20. *Let* $C \in W_\Theta \backslash W$ *and* $\alpha \in \Pi \cap \Pi_\lambda$ *so that* $Cs_\alpha < C$*. Then*

$$I_{s_\alpha}\mathcal{L}(w^C s_\alpha, \lambda, \eta) \neq 0, \text{ and}$$
$$L^{-1}I_{s_\alpha}\mathcal{L}(w^C s_\alpha, \lambda, \eta) = 0.$$

We are ready to complete the proof of Theorem 3.1.8.

Proof of 3.1.8(b)(c). The exact sequence (3.1.12) for $\mathcal{V} = \mathcal{L}(ws_\alpha, \lambda)$ says

$$0 \longrightarrow H^{-1}U\mathcal{L}(ws_\alpha, \lambda) \longrightarrow L^{-1}I_{s_\alpha}\mathcal{L}(ws_\alpha, \lambda) \longrightarrow 0.$$

By the preceding proposition 3.1.20, the second term vanishes. By exactness, the first term also vanishes.

For $H^1U\mathcal{L}(ws_\alpha, \lambda)$, we look at the sequence (3.1.13):

$$0 \longrightarrow H^0U\mathcal{L}(ws_\alpha, \lambda) \longrightarrow I_{s_\alpha}\mathcal{L}(ws_\alpha, \lambda) \longrightarrow \mathcal{L}(ws_\alpha, s_\alpha\lambda) \longrightarrow H^1U\mathcal{L}(ws_\alpha, \lambda) \longrightarrow 0. \qquad (3.1.21)$$

Suppose $H^1U\mathcal{L}(ws_\alpha, \lambda) \neq 0$, then the last map in the sequence must be an isomorphism because $\mathcal{L}(ws_\alpha, s_\alpha\lambda)$ is irreducible. In particular, $H^1U\mathcal{L}(ws_\alpha, \lambda)$ is supported on $\overline{C(ws_\alpha)}$, and any fiber of p_α that meets $C(ws_\alpha)$ intersects it at a single point. On the other hand, recall that $H^1U\mathcal{L}(ws_\alpha, \lambda)$ is a twist of $H^1U_\alpha\mathcal{L}(ws_\alpha, \lambda)$. By 3.1.10, any fiber of p_α that meets its support must be fully contained in the support. This is a contradiction. Thus $H^1U\mathcal{L}(ws_\alpha, \lambda)$ is also zero. This proves part (b).

It remains to prove part (c). From the calculation (3.1.19), we see that the support of $I_{s_\alpha}\mathcal{L}(ws_\alpha, \lambda)$ contains $C(w)$. From the definition of LI_{s_α}, the support of $I_{s_\alpha}\mathcal{L}(ws_\alpha, \lambda)$ is contained in $\overline{C(w)}$. So the support equals $\overline{C(w)}$. In view of the sequence (3.1.21), this forces the support of $H^0U\mathcal{L}(ws_\alpha, \lambda)$ to also equal to $\overline{C(w)}$ since $\mathcal{L}(ws_\alpha, s_\alpha\lambda)$ is supported in a subset with strictly smaller dimension. On the other hand, we know from part (a) that $H^0U\mathcal{L}(ws_\alpha, \lambda)$ is a direct sum of irreducible modules. So it contains a direct summand supported on $\overline{C(w)}$. This summand can only be $\mathcal{L}(w, s_\alpha\lambda)$. Identifying U with a twist of U_α, we see that $\mathcal{L}(w, \lambda)$ is a direct summand of $H^0U_\alpha\mathcal{L}(ws_\alpha, \lambda)$. This proves (c) and completes the proof of 3.1.8. □

3.2 Non-integral intertwining functors

In this section, we study I_{s_β} for a non-integral simple root β.

We will use the following easy fact in a number of occasions which is mentioned in the introduction and is proven in [MS14]. We include a proof for completeness. Recall that $\Theta \subset \Pi$ is defined to be the subset of simple roots α so that $\eta|_{\mathfrak{g}_\alpha} \neq 0$, and ${}^\Theta W$ is a cross-section of $W_\Theta \backslash W$ consisting of the unique longest elements in each coset.

Lemma 3.2.1.

(a) If $w \notin {}^{\Theta}W$, $\mathrm{Mod}_{coh}(\mathcal{D}_{C(w)}, N, \eta) = 0$;

(b) If $w \in {}^{\Theta}W$, $\mathrm{Mod}_{coh}(\mathcal{D}_{C(w)}, N, \eta)$ *is semisimple with a unique irreducible object, denoted by* $\mathcal{O}^{\eta}_{C(w)}$, *which has* $\mathcal{O}_{C(w)}$ *as its underlying* $\mathcal{O}$*-module.*

Proof. Let $x_w \in C(w)$, write $N_w = \mathrm{Stab}_N(x_w)$ for the stabilizer, and write $\mathfrak{n}_w$ for its Lie algebra. Then $C(w) = N \times_{N_w} \{x_w\}$. So by descent,

$$\mathrm{Mod}_{coh}(\mathcal{D}_{C(w)}, N, \eta) \cong \mathrm{Mod}_{coh}(\mathbb{C}, N_w, \eta).$$

Here, since $N_w \circlearrowright \{x_w\}$ trivially, $\mathbb{C} = \mathcal{D}_{\{x_w\}}$ naturally comes with the trivial action of N_w and the trivial map $0 : \mathfrak{n}_w \to \mathbb{C}$. By definition, an object on the right side is a finite dimensional vector space V equipped with a linear action $N_w \circlearrowright_{grp} V$ so that the differential $\mathfrak{n}_w \circlearrowright_{grp} V$ of the N_w-action differs from the action $\mathfrak{n}_w \xrightarrow{0} \mathbb{C} \circlearrowright V$ by $-\eta|_{\mathfrak{n}_w}$, that is, $\mathfrak{n}_w \circlearrowright_{grp} V$ is given by $-\eta|_{\mathfrak{n}_w}$. Since N_w is unipotent, $\mathfrak{n}_w \circlearrowright_{grp} V$ must be nilpotent. On the other hand, the action of any element $\xi \in \mathfrak{n}_w$ given by $-\eta|_{\mathfrak{n}_w}$ is semisimple. Hence, if V is nonzero, $-\eta|_{\mathfrak{n}_w}$ must be zero and $\mathfrak{n}_w \circlearrowright_{grp} V$ must be trivial.

The roots in $\mathfrak{n}_w$ are $\Sigma^+ \cap w\Sigma^+$. Hence $\eta|_{\mathfrak{n}_w} = 0$ is equivalent to $\Theta \cap w\Sigma^+ = \varnothing$, or equivalently $w^{-1}\Theta \subseteq -\Sigma^+$, i.e. $w \in {}^{\Theta}W$. So $\mathrm{Mod}_{coh}(\mathbb{C}, N_w, \eta)$ contains a nonzero object only if $w \in {}^{\Theta}W$.

Suppose $w \in {}^{\Theta}W$ and $V \in \mathrm{Mod}_{coh}(\mathbb{C}, N_w, \eta)$ is nonzero. We have seen that $\mathfrak{n}_w \circlearrowright_{grp} V$ is trivial. Hence $N_w \circlearrowright_{grp} V$ is also trivial.[2] So the category $\mathrm{Mod}_{coh}(\mathbb{C}, N_w, \eta)$ is just the category of finite dimensional vector spaces, which is semisimple with a unique irreducible object. Inducing to N, we see that $\mathrm{Mod}_{coh}(\mathcal{D}_{C(w)}, N, \eta)$ is semisimple with a unique irreducible object whose underlying $\mathcal{O}$-module is $\mathcal{O}_{C(w)}$. □

To use the intertwining functors for our purpose, we need to compute the action of intertwining functors on standard and irreducible modules. Romanov computed the following result for $Cs_\beta > C$. The main ingredients of the proof there are base change formula and projection formula for $\mathcal{D}$-modules. We reproduce the argument here.

As in the previous section, η will always be fixed, and we will omit writing η and line bundle twists in the proofs.

[2]The representation $\mathfrak{n}_w \to \mathfrak{gl}(\mathcal{F}(x_w))$ is the differentiation of $N_w \to \mathbf{GL}(\mathcal{F}(x_w))$. Since the firs map is trivial, the image of the second map must be a finite subgroup. But N_w is connected, so the image must be connected. This forces the image of the second map to be trivial.

Proposition 3.2.2 ([Rom21, Proposition 6])**.** *Let $\beta \in \Pi$ and $C \in W_\Theta \backslash W$ such that $Cs_\beta > C$. Then for any $\lambda \in \mathfrak{h}^*$,*

$$LI_{s_\beta}\mathcal{I}(w^C, \lambda, \eta) = \mathcal{I}(w^C s_\beta, s_\beta\lambda, \eta).$$

Proof. Recall the variety $Z_{s_\alpha} \subset X \times X$ defined at the beginning of the previous section, with projection maps $p_1, p_2 : Z_{s_\alpha} \to X$. Since $Cs_\beta > C$, the preimage of $C(w^C)$ under p_2 is isomorphic to $C(w^C s_\beta)$ via p_1. So we have the following diagram

$$\begin{array}{ccccc} C(w^C s_\beta) & = & C(w^C s_\beta) & \xrightarrow{\varpi_2} & C(w^C) \\ \downarrow{\scriptstyle i_{w^C s_\beta}} & & \downarrow{\scriptstyle \tilde{i}_{w^C s_\beta}} & & \downarrow{\scriptstyle i_{w^C}} \\ X & \xleftarrow{p_1} & Z_{s_\alpha} & \xrightarrow{p_2} & X \end{array}$$

where the right square is Cartesian. Thus, by base change,

$$\begin{aligned} LI_{s_\beta}\mathcal{I}(w^C, \lambda) &= p_{1+}p_2^+ i_{w^C +}\mathcal{O}^\eta_{C(w^C)} \\ &= p_{1+}\tilde{i}_{w^C s_\beta +}\varpi_2^+\mathcal{O}^\eta_{C(w^C)} \\ &= i_{w^C s_\beta +}\mathcal{O}^\eta_{C(w^C s_\beta)} \\ &= \mathcal{I}(w^C s_\beta, s_\beta\lambda). \end{aligned}$$

□

Combined with 3.1.2, this implies

Corollary 3.2.3. *Let $\beta \in \Pi - \Pi_\lambda$ and $C \in W_\Theta \backslash W$ such that $Cs_\beta \neq C$. Then*

$$I_{s_\beta}\mathcal{I}(w^C, \lambda, \eta) = \mathcal{I}(w^C s_\beta, s_\beta\lambda, \eta).$$

Proof. Suppose $Cs_\beta > C$, then the statement follows from 3.2.2. But since I_{s_β} is an equivalence of categories with inverse I_{s_β},

$$\mathcal{I}(w^C, \lambda) = I_{s_\beta}\mathcal{I}(w^C s_\beta, s_\beta\lambda).$$

□

It remains to consider the case $Cs_\beta = C$. This case requires a bit more care than the previous case since the preimage of $C(w^C)$ under p_2 no longer has a very clean description.

Recall that, for a simple root β, $p_\beta : X \to X_\beta$ is the natural projection to the partial flag variety of type β. This is a Zariski-local $\mathbb{A}^1$-fibration. x and y are contained in the same p_β-fiber (i.e. $p_\beta(x) = p_\beta(y)$) if and only if $\mathfrak{b}_x$ and $\mathfrak{b}_y$ are in relative position 1 (i.e. $\mathfrak{b}_x = \mathfrak{b}_y$) or s_β.

Lemma 3.2.4. *Let $C \in W_\Theta \backslash W$ and $\beta \in \Pi$ such that $Cs_\beta = C$. Set*

$$S = \{(x, y) \in C(w^C) \times C(w^C) \mid \mathfrak{b}_x \textit{ and } \mathfrak{b}_y \textit{ are in relative position } s_\beta\} \subset Z_{s_\beta}.$$

Write $C(w^C) \xleftarrow{p_1|_S} S \xrightarrow{p_2|_S} C(w^C)$ for the projections. Then

$$(p_1|_S)_+(p_2|_S)^+\mathcal{O}^\eta_{C(w^C)} = \mathcal{O}^\eta_{C(w^C)}.$$

Proof. For convenience, write $w = w^C$, $p_1 = p_1|_S$ and $p_2 = p_2|_S$. Set

$$S' = C(w) \times_{p_\beta(C(w))} C(w) = \{(x,y) \in C(w) \times C(w) \mid p_\beta(x) = p_\beta(y)\}.$$

Then $S \subset S \cup \Delta_{C(w)} = S' \subset Z_{s_\beta}$, where $\Delta_{C(w)}$ denotes the diagonal. Write $C(w) \xleftarrow{q_1} S' \xrightarrow{q_2} C(w)$ for the projections, and $\Delta_{C(w)} \xrightarrow{i_\Delta} S' \xleftarrow{i_S} S$ for the inclusions. Then i_Δ is a closed immersion with relative dimension 1, and i_S is open. We have the following diagram

$$\begin{array}{ccccc} S & & & & \\ & \searrow^{i_S} & & \overset{p_2}{\curvearrowright} & \\ & & S' & \xrightarrow{q_2} & C(w) \\ & {}_{p_1}\!\!\searrow & \downarrow{\scriptstyle q_1} & & \downarrow{\scriptstyle p_\beta} \\ & & C(w) & \xrightarrow{p_\beta} & p_\beta(C(w)) \end{array} \tag{3.2.5}$$

where the bottom-right square is Cartesian.

Applying the distinguished triangle for the immersions i_Δ and i_S to $q_2^+\mathcal{O}^\eta_{C(w)}$, we get

$$\begin{array}{ccc} i_{\Delta+}i_\Delta^! q_2^+\mathcal{O}^\eta_{C(w)} & \longrightarrow & q_2^+\mathcal{O}^\eta_{C(w)} \\ {\scriptstyle [1]}\nwarrow & & \swarrow \\ & i_{S+}i_S^+ q_2^+\mathcal{O}^\eta_{C(w)} & \end{array} .$$

Applying q_{1+}, we get

$$\begin{array}{ccc} q_{1+}i_{\Delta+}i_\Delta^! q_2^+\mathcal{O}^\eta_{C(w)} & \longrightarrow & q_{1+}q_2^+\mathcal{O}^\eta_{C(w)} \\ {\scriptstyle [1]}\nwarrow & & \swarrow \\ & q_{1+}i_{S+}i_S^+ q_2^+\mathcal{O}^\eta_{C(w)} & \end{array} .$$

Applying base change to the bottom-right square in (3.2.5), $q_{1+}q_2^+\mathcal{O}_{C(w)} \cong p_\beta^+ p_{\beta+}\mathcal{O}^\eta_{C(w)}$. Here $p_{\beta+}\mathcal{O}^\eta_{C(w)}$ is an η-twisted Harish-Chandra sheaf on $p_\beta(C(w))$. But $p_\beta(C(w))$ is isomorphic to $C(ws_\beta)$ as an N-variety via p_β, and since ws_β is not the longest element in $W_\Theta w s_\beta = W_\Theta w$, we know there is no η-twisted Harish-Chandra sheaf on $C(ws_\beta)$ except 0. Hence $p_{\beta+}\mathcal{O}^\eta_{C(w)} = 0$ and thus $q_{1+}q_2^+\mathcal{O}_{C(w)} = 0$. As a result,

$$q_{1+}i_{S+}i_S^+ q_2^+\mathcal{O}^\eta_{C(w)} = q_{1+}i_{\Delta+}i_\Delta^! q_2^+\mathcal{O}^\eta_{C(w)}[1].$$

The left side simplifies to $p_{1+}p_2^+\mathcal{O}^\eta_{C(w)}$. For the right side, $q_{1+}i_{\Delta+} = (q_1 \circ i_\Delta)_+$ and $q_1 \circ i_\Delta$ is the projection $\Delta_{C(w)} \to C(w)$ along the first coordinate which is an N-equivariant isomorphism. Moreover,

$$\begin{aligned} i_\Delta^! q_2^+\mathcal{O}^\eta_{C(w)}[1] &= i_\Delta^+ q_2^+\mathcal{O}^\eta_{C(w)}[1][-1] \\ &= (q_2 \circ i_\Delta)^+\mathcal{O}^\eta_{C(w)}, \end{aligned}$$

and $q_2 \circ i_\Delta$ is the projection $\Delta_{C(w)} \to C(w)$ along the second coordinate, also an N-equivariant isomorphism. Thus

$$p_{1+}p_2^+\mathcal{O}^\eta_{C(w)} = (q_1 \circ i_\Delta)_+(q_2 \circ i_\Delta)^+\mathcal{O}^\eta_{C(w)} = \mathcal{O}^\eta_{C(w)}. \qquad \square$$

Lemma 3.2.6. *Let* $s_\beta \in \Pi$ *and* $C \in W_\Theta\backslash W$ *such that* $Cs_\beta = C$. *Write* $\iota : C(w^C) \hookrightarrow C(w^C) \cup C(w^C s_\beta)$ *for the inclusion. Then for any* $\mathcal{F} \in \mathrm{Mod}_{coh}(\mathcal{D}_{C(w^C)\cup C(w^C s_\beta)}, N, \eta)$,

$$\mathcal{F} = \iota_+\iota^!\mathcal{F} = (\iota_+\mathcal{O}^\eta_{C(w^C)})^{\oplus \operatorname{rank} \iota^!\mathcal{F}}$$

where rank *stands for the rank as a free* $\mathcal{O}$*-module.*

Proof. Write $w = w^C$. The assumption implies that $ws_\beta \in C$, $ws_\beta < w$, and that $C(w)$ and $C(ws_\beta)$ are open and closed in $C(w) \cup C(ws_\beta)$, respectively.

Since the category of η-twisted Harish-Chandra sheaves on $C(w)$ is semisimple, $\iota^!\mathcal{F}$ is a direct sum of copies of $\mathcal{O}^\eta_{C(w)}$. This implies the second equality. For the first equality, adjunction gives a map

$$\mathcal{F} \to \iota_+\iota^!\mathcal{F} \tag{3.2.7}$$

whose kernel and cokernel are supported on $C(ws_\beta)$, which are equal to direct images of η-twisted Harish-Chandra sheaves on $C(ws_\beta)$ by Kashiwara's equivalence. But ws_β is not the longest element in C, so there is no such module on $C(ws_\beta)$ except zero. Hence (3.2.7) is an isomorphism, which establishes the first equality. $\square$

Proposition 3.2.8. *Let* $C \in W_\Theta\backslash W$, $\beta \in \Pi$ *such that* $Cs_\beta = C$. *Then*

$$LI_{s_\beta}\mathcal{I}(w^C, \lambda, \eta) = \mathcal{I}(w^C, s_\beta\lambda, \eta).$$

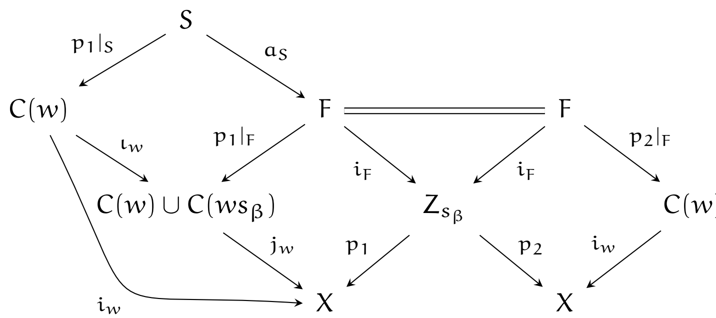
Proof. Write $w = w^C$. Let

$$F = Z_{s_\beta} \times_{p_2, X, i_w} C(w) = \{(x, y) \in X \times C(w) \mid \mathfrak{b}_x \text{ and } \mathfrak{b}_y \text{ are in relative position } s_\beta\}.$$

and let S be as in 3.2.4. Then S is a subvariety of F. It's easy to see that

$$\begin{aligned} p_1(F) &= \{x \in X \mid \exists y \in C(w) \text{ such that } \mathfrak{b}_x \text{ and } \mathfrak{b}_y \text{ are in relative position } s_\beta\} \\ &= C(w) \cup C(ws_\beta). \end{aligned}$$

So we have the following diagram



(3.2.9)

The right-most square is Cartesian by definition of F. The top-left square is also Cartesian, i.e. S is the preimage of $C(w)$ along $p_1 : F \to C(w) \cup C(ws_\beta)$. By definition of intertwining functors and base change,

$$\begin{aligned} LI_{s_\beta}\mathcal{I}(w,\lambda) &= p_{1+}p_2^+ i_{w+}\mathcal{O}^\eta_{C(w)} \\ &= p_{1+}i_{F+}(p_2|_F)^+\mathcal{O}^\eta_{C(w)} \\ &= j_{w+}(p_1|_F)_+(p_2|_F)^+\mathcal{O}^\eta_{C(w)}. \end{aligned} \tag{3.2.10}$$

We claim that $(p_1|_F)_+(p_2|_F)^+\mathcal{O}^\eta_{C(w)} = \iota_{w+}\mathcal{O}^\eta_{C(w)}$. By 3.2.6,

$$(p_1|_F)_+(p_2|_F)^+\mathcal{O}^\eta_{C(w)} = \iota_{w+}\iota_w^!(p_1|_F)_+(p_2|_F)^+\mathcal{O}^\eta_{C(w)}.$$

Apply base change using the top-left square in (3.2.9),

$$\iota_w^!(p_1|_F)_+(p_2|_F)^+\mathcal{O}^\eta_{C(w)} = (p_1|_S)_+a_S^!(p_2|_F)^+\mathcal{O}^\eta_{C(w)} = (p_1|_S)_+a_S^+(p_2|_F)^+\mathcal{O}^\eta_{C(w)}$$

Note that $p_2|_F \circ a_S = p_2|_S$. Hence, by 3.2.4, the sheaf in the above equation equals $\mathcal{O}^\eta_{C(w)}$. This proves the claim. As a result,

$$\begin{aligned} (3.2.10) &= j_{w+}\iota_{w+}\mathcal{O}^\eta_{C(w)} \\ &= i_{w+}\mathcal{O}^\eta_{C(w)} \\ &= \mathcal{I}(w, s_\beta\lambda, \eta) \end{aligned}$$

which proves the proposition. □

Corollary 3.2.11. *Let $\beta \in \Pi - \Pi_\lambda$. Let $C \in W_\Theta \backslash W$. Then*

$$I_{s_\beta}\mathcal{I}(w^C, \lambda, \eta) = \mathcal{I}(w^{Cs_\beta}, s_\beta\lambda, \eta),$$

$$I_{s_\beta}\mathcal{L}(w^C, \lambda, \eta) = \mathcal{L}(w^{Cs_\beta}, s_\beta\lambda, \eta)$$

(note that we have w^{Cs_β} instead of $w^C s_\beta$ on the right hand sides).

Proof. The statement about standard modules is the combination of 3.2.3 and 3.2.8. Since I_{s_β} is an equivalence of categories, it must send the unique irreducible submodule of $\mathcal{I}(w^C,\lambda,\eta)$ to the unique irreducible submodule of $\mathcal{I}(w^{Cs_\beta},s_\beta\lambda,\eta)$, i.e. it must send $\mathcal{L}(w^C,\lambda,\eta)$ to $\mathcal{L}(w^{Cs_\beta},s_\beta\lambda,\eta)$. □

Next, we show that non-integral intertwining functors also preserves pullback of irreducible modules to strata.

Proposition 3.2.12. *Let* $\beta \in \Pi - \Pi_\lambda$, $C, D \in W_\Theta\backslash W$ *and* $p \in \mathbb{Z}$. *Then*

$$\operatorname{rank} H^p i^!_{w^D}\mathcal{L}(w^C,\lambda,\eta) = \operatorname{rank} H^p i^!_{w^{Ds_\beta}}\mathcal{L}(w^{Cs_\beta},s_\beta\lambda,\eta).$$

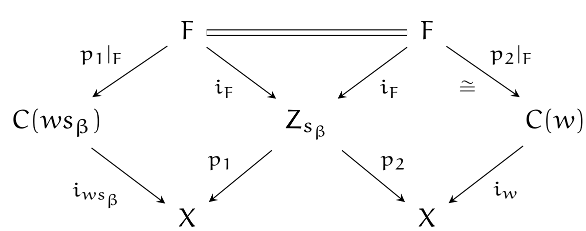
The proof we give below uses the same tools as in the previous proposition. There is an alternative proof which we briefly mention. One can show that $\operatorname{rank} H^p i^!_{w^D}\mathcal{L}(w^C,\lambda,\eta)$ equals the dimension of the p-th $\mathcal{D}_\lambda$-module Ext group of $\mathcal{M}(w^C,\lambda,\eta)$ and $\mathcal{L}(w^C,\lambda,\eta)$ using facts on derived categories of highest weight categories (Brown-Romanov [BR22, Theorem 7.2] showed that $\operatorname{Mod}_{coh}(\mathcal{D}_\lambda,N,\eta)$ is a highest weight category). The proposition follows from the fact that I_{s_β} is an equivalence of categories and induces an isomorphism on Ext-groups.

Proof. Write $w = w^D$.

There are two cases, $Ds_\beta \neq D$ or $Ds_\beta = D$. Consider the first case. Assume $Ds_\beta < D$. Then $w^{Ds_\beta} = w^D s_\beta = ws_\beta$. Let

$$F = C(ws_\beta)\times_{i_{ws_\beta},X,p_1} Z_{s_\beta} = \{(x,y)\in C(ws_\beta)\times X \mid \mathfrak{b}_x \text{ and } \mathfrak{b}_y \text{ are in relative position } s_\beta\}.$$

Then the second projection $p_2|_F : F \to X$ induces an isomorphism of F onto $C(w)$, and we have the following commuting diagram



where the left square is Cartesian. Using the preceding proposition and base change,

$$\begin{aligned}\operatorname{rank} H^p i^!_{w^{Ds_\beta}}\mathcal{L}(w^{Cs_\beta},s_\beta\lambda,\eta) &= \operatorname{rank} H^p i^!_{ws_\beta} I_{s_\beta}\mathcal{L}(w^C,\lambda,\eta)\\ &= \operatorname{rank} H^p i^!_{ws_\beta} p_{1+}p_2^+\mathcal{L}(w^C,\lambda,\eta) \qquad (3.2.13)\\ &= \operatorname{rank} H^p (p_1|_F)_+(p_2|_F)^! i^!_w\mathcal{L}(w^C,\lambda,\eta)[-1]. \qquad (3.2.14)\end{aligned}$$

Since $\mathrm{Mod}_{coh}(\mathcal{D}_{C(w)}, N, \eta)$ is semisimple, $i_w^! \mathcal{L}(w^C, \lambda, \eta)$ is a direct sum of $\mathcal{O}^\eta_{C(w)}$'s at different degrees. So $(p_2|_F)^! i_w^! \mathcal{L}(w^C, \lambda, \eta)$ is a direct sum of $\mathcal{O}^\eta_F$'s at different degrees ($p_2|_F$ is an isomorphism), and the rank at a degree is the same as that of $i_w^! \mathcal{L}(w^C, \lambda, \eta)$. It is therefore enough to compute $(p_1|_F)_+ \mathcal{O}^\eta_F$. For this, we use the fact that a map of homogeneous spaces of a unipotent group is isomorphic to a coordinate projection of affine spaces, that is, we have the following commutative diagram where all maps are N-equivariant, for some N-actions on $\mathbb{A}^1 \times \mathbb{A}^{\ell(ws_\beta)}$ and $\mathbb{A}^{\ell(ws_\beta)}$:

$$\begin{array}{ccc} F & \xrightarrow{p_1|_F} & C(ws_\beta) \\ \cong\downarrow & & \downarrow\cong \\ \mathbb{A}^1 \times \mathbb{A}^{\ell(ws_\beta)} & \xrightarrow{pr_1} & \mathbb{A}^{\ell(ws_\beta)} \end{array} .$$

So it suffices to compute $pr_{1+} \mathcal{O}^\eta_{\mathbb{A}^1 \times \mathbb{A}^{\ell(ws_\beta)}}$. Since pr_1 is a coordinate projection, $pr_1^+ \mathcal{O}^\eta_{\mathbb{A}^{\ell(ws_\beta)}} = \mathcal{O}_{\mathbb{A}^1} \boxtimes \mathcal{O}^\eta_{\mathbb{A}^{\ell(ws_\beta)}}$ (we remark that, without the assumption of $Ds_\beta \neq D$, ws_β and w can be in the same right W_Θ-coset, in which case $\mathcal{O}^\eta_{\mathbb{A}^{\ell(ws_\beta)}}$ does not exist). On the other hand, $pr_1^+ \mathcal{O}^\eta_{\mathbb{A}^{\ell(ws_\beta)}}$ is a rank one η-twisted sheaf, which can only be $\mathcal{O}^\eta_{\mathbb{A}^1 \times \mathbb{A}^{\ell(ws_\beta)}}$. We conclude that

$$\mathcal{O}^\eta_{\mathbb{A}^1 \times \mathbb{A}^{\ell(ws_\beta)}} = \mathcal{O}_{\mathbb{A}^1} \boxtimes \mathcal{O}^\eta_{\mathbb{A}^{\ell(ws_\beta)}}.$$

As a result, writing $p : \mathbb{A}^1 \to \{*\}$ for the unique morphism to a point,

$$\begin{aligned} pr_{1+} \mathcal{O}^\eta_{\mathbb{A}^1 \times \mathbb{A}^{\ell(ws_\beta)}} &= (p_+ \mathcal{O}_{\mathbb{A}^1}) \boxtimes \left((\mathrm{Id}_{\mathbb{A}^{\ell(ws_\beta)}})_+ \mathcal{O}^\eta_{\mathbb{A}^{\ell(ws_\beta)}}\right) \\ &= \mathbb{C}[1] \boxtimes \mathcal{O}^\eta_{\mathbb{A}^{\ell(ws_\beta)}} \\ &= \mathcal{O}_{\mathbb{A}^{\ell(ws_\beta)}}[1]. \end{aligned}$$

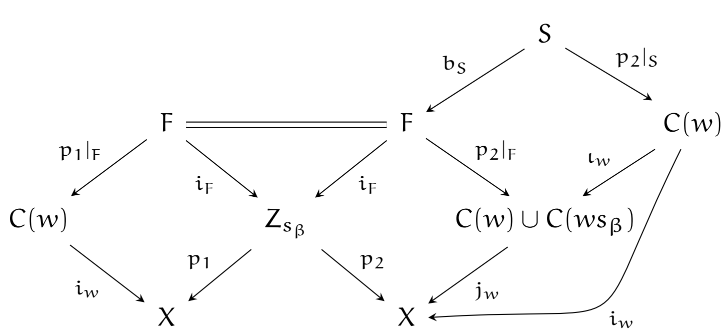
Therefore $(p_1|_F)_+ \mathcal{O}^\eta_F = \mathcal{O}^\eta_{C(ws_\beta)}[1]$ and hence

$$\operatorname{rank} H^p i^!_{w^{Ds_\beta}} \mathcal{L}(w^{Cs_\beta}, s_\beta\lambda, \eta) = (3.2.14) = \operatorname{rank} H^p i_w^! \mathcal{L}(w^C, \lambda, \eta).$$

Now consider the case $Ds_\beta = D$. In this case $w^{Ds_\beta} = w^D = w$. Set

$$F = C(w) \times_{i_w, X, p_1} Z_{s_\beta} = \{(x, y) \in C(w) \times X \mid \mathfrak{b}_x \text{ and } \mathfrak{b}_y \text{ are in relative position } s_\beta\}$$

and set S as in 3.2.4, viewed as a subvariety of F. Then the following diagram commutes



where the leftmost square and the top-right square are Cartesian. By the preceding proposition and base change,

$$\begin{aligned} i^!_{w^D s_\beta} \mathcal{L}(w^C s_\beta, s_\beta \lambda, \eta) &= i^!_w I_{s_\beta} \mathcal{L}(w^C, \lambda, \eta) \\ &= (p_1|_F)_+ (p_2|_F)^! j^!_w \mathcal{L}(w^C, \lambda, \eta)[-1]. \end{aligned} \tag{3.2.15}$$

By 3.2.6, $j^!_w \mathcal{L}(w^C, \lambda, \eta) = \iota_{w+} \iota^!_w j^!_w \mathcal{L}(w^C, \lambda, \eta)$. Hence

$$\begin{aligned} (3.2.15) &= (p_1|_F)_+ (p_2|_F)^! \iota_{w+} \iota^!_w j^!_w \mathcal{L}(w^C, \lambda, \eta)[-1] \\ &= (p_1|_F)_+ b_{S+} (p_2|_S)^+ i^!_w \mathcal{L}(w^C, \lambda, \eta). \end{aligned} \tag{3.2.16}$$

Here $p_1|_F \circ b_S = p_1|_S$. Also $i^!_w \mathcal{L}(w^C, \lambda, \eta)$ is a direct sum of $\mathcal{O}^\eta_{C(w)}$ in various degrees. Hence by 3.2.4,

$$\operatorname{rank} H^p i^!_{w^D s_\beta} \mathcal{L}(w^C s_\beta, s_\beta \lambda, \eta) = \operatorname{rank} H^p (3.2.16) = \operatorname{rank} H^p i^!_w \mathcal{L}(w^C, \lambda, \eta). \qquad \square$$

Chapter 4

Main algorithm

In this section, we formulate and prove an algorithm for computing a set of polynomials in q indexed by pairs of right W_Θ-cosets whose evaluation at $q = -1$ leads to the character formula for irreducible modules. This is in the same spirit as the ordinary Kazhdan-Lusztig algorithm for category $\mathcal{O}$. The algorithm we will prove is suggested by Miličić and is modified from the ones in [Rom21], [Mil].

In §4.1, we define the Whittaker Kazhdan-Lusztig polynomials, the module $\mathcal{H}_\Theta$, and related notations. The statement of the algorithm is contained in §4.2. Proof of the algorithm is divided into sections that follow.

4.1 Whittaker Kazhdan-Lusztig polynomials

In this section we define the Whittaker Kazhdan-Lusztig polynomials. Since there already exists a vast literature on Kazhdan-Lusztig theory, I have chosen not to present too much background. Readers can look at [Rom21, §6] for comparisons with other versions of Kazhdan-Lusztig polynomials.

Recall the sets $A_{\Theta,\lambda} \subseteq W$ and $\Theta(u,\lambda) \subseteq \Pi_\lambda$ ($u \in A_{\Theta,\lambda}$) defined in §2.3 and §2.4. Recall also that we have a partial order on $W_\Theta \backslash W$ inherited from the Bruhat order on ${}^\Theta W$, denoted by $\leqslant$. Similarly, we have a partial order on $W_{\lambda,\Theta(u,\lambda)} \backslash W_\lambda$ which we denote by $\leqslant_{u,\lambda}$. By our convention 2.4.5, $C \leqslant_{u,\lambda} D$ means that C and D are both in $W_\Theta u W_\lambda$ and $C|_\lambda \leqslant_{u,\lambda} D|_\lambda$. So, $C \nleqslant_{u,\lambda} D$ means either C and D are not in the same (W_Θ, W_λ)-coset, or they are in the same coset $W_\Theta u W_\lambda$ but $C|_\lambda \nleqslant_{u,\lambda} D|_\lambda$. Here $(-)|_\lambda$ is the bijection

$$(-)|_\lambda : W_\Theta \backslash W \to \bigcup_{u \in A_{\Theta,\lambda}} W_{\lambda,\Theta(u,\lambda)} \backslash W_\lambda$$

defined in 2.4.5.

Let $\mathcal{H}_\Theta$ be the free $\mathbb{Z}[q,q^{-1}]$-modules with basis δ_C, $C \in W_\Theta\backslash W$. For any $\alpha \in \Pi$, define a $\mathbb{Z}[q,q^{-1}]$-linear operator on $\mathcal{H}_\Theta$ by

$$T_\alpha(\delta_C) = \begin{cases} q\delta_C + \delta_{Cs_\alpha} & \text{if } Cs_\alpha > C; \\ 0 & \text{if } Cs_\alpha = C; \\ q^{-1}\delta_C + \delta_{Cs_\alpha} & \text{if } Cs_\alpha < C. \end{cases}$$

The module $\mathcal{H}_\Theta$ is a module of the full Hecke algebra $\mathcal{H}$, and T_α encodes the action of the Kazhdan-Lusztig basis elements in $\mathcal{H}$.

For an element u in $A_{\Theta,\lambda}$, let $\mathcal{H}_{\Theta(u,\lambda)}$ be the free $\mathbb{Z}[q,q^{-1}]$-module with basis δ_E, $E \in W_{\lambda,\Theta(u,\lambda)}\backslash W_\lambda$. Define the operator $T_\alpha^{u,\lambda}$ in the same way as T_α, replacing $\alpha \in \Pi$ by $\alpha \in \Pi_\lambda$, C by E, and $>$, $<$ by $>_{u,\lambda}$, $<_{u,\lambda}$, respectively. Namely,

$$T_\alpha^{u,\lambda}(\delta_E) = \begin{cases} q\delta_E + \delta_{Es_\alpha} & \text{if } Es_\alpha >_{u,\lambda} E; \\ 0 & \text{if } Es_\alpha = E; \\ q^{-1}\delta_E + \delta_{Es_\alpha} & \text{if } Es_\alpha <_{u,\lambda} E. \end{cases}$$

We will use a left action of W on $\mathcal{H}_\Theta$ defined by $w \cdot \delta_C = \delta_{wC}$. Similarly, a right action of W on $\mathcal{H}_\Theta$ is defined by $\delta_C \cdot w = \delta_{Cw}$. We will simply write $w\delta_C$, $\delta_C w$ for the actions, omitting the dots. $w(-)w^{-1}$ then denotes the simultaneous action of w on the left and w^{-1} on the right. By 2.4.8, $s_\beta(-)s_\beta$ defines a bijection

$$s_\beta(-)s_\beta : W_{\lambda,\Theta(u,\lambda)}\backslash W_\lambda \xrightarrow{\sim} W_{s_\beta\lambda,\Theta(r,s_\beta\lambda)}\backslash W_{s_\beta\lambda}$$

where $r \in A_{\Theta,s_\beta\lambda}$ is the unique element representing the coset $W_\Theta u s_\beta W_{s_\beta\lambda}$. We extend this to an isomorphism

$$s_\beta(-)s_\beta : \mathcal{H}_{\Theta(u,\lambda)} \xrightarrow{\sim} \mathcal{H}_{\Theta(r,s_\beta\lambda)}, \quad \delta_E \mapsto \delta_{s_\beta E s_\beta}.$$

We also extend $(-)|_\lambda$ to a map

$$(-)|_\lambda : \mathcal{H}_\Theta \xrightarrow{\sim} \bigoplus_{u\in A_{\Theta,\lambda}} \mathcal{H}_{\Theta(u,\lambda)}, \quad \delta_C \mapsto \delta_{C|_\lambda}.$$

The following theorem, proven in [Rom21, Theorem 11], defines a set of polynomials indexed by pairs of right cosets, called **Whittaker Kazhdan-Lusztig polynomials**. It is verified in *op. cit.* that these polynomials are dual to the parabolic Kazhdan-Lusztig polynomials. More details of these comparisons can be found in *op. cit.* For a right coset $E \in W_\Theta\backslash W$, we write $\left(W_\Theta\backslash W\right)_{\leqslant E}$ for the set of those cosets F such that $F \leqslant E$.

Whittaker Kazhdan-Lusztig polynomials for (W,Π,Θ) **4.1.1.** *For any* $E \in W_\Theta\backslash W$*, there exists a unique set of polynomials* $\{P_{CD}\} \subset q\mathbb{Z}[q]$ *indexed by*

$$\left\{(C,D) \mid C,D \in \left(W_\Theta\backslash W\right)_{\leqslant E}; D < C\right\}$$

such that the function

$$\psi : \left(W_\Theta \backslash W\right)_{\leqslant E} \longrightarrow \mathcal{H}_\Theta, \quad C \mapsto \delta_C + \sum_{D<C} P_{CD}\delta_D$$

satisfies the following property: for any $C \in W_\Theta\backslash W$ *with* $C \neq W_\Theta$*, there exist* $\alpha \in \Pi$ *and* $c_D \in \mathbb{Z}$ *such that* $Cs_\alpha < C$ *and*

$$T_\alpha(\psi(Cs_\alpha)) = \sum_{D\leqslant C} c_D \psi(D).$$

Moreover, the polynomials P_{CD} *do not depend on the choice of* E*, and they satisfy the parity condition*

$$P_{CD} \in \mathbb{Z}[q^2, q^{-2}]q^{\ell(C)-\ell(D)}.$$

The elements $\psi(C)$*'s will be referred to as (Whittaker) Kazhdan-Lusztig basis elements.* [1]

Here $\ell(C)$ means $\ell(w^C)$.

We apply the same definition to $(W_\lambda, \Pi_\lambda, \Theta(u, \lambda))$:

Whittaker Kazhdan-Lusztig polynomials for $(W_\lambda, \Pi_\lambda, \Theta(u, \lambda))$ **4.1.2.** *For any* $E \in W_{\lambda,\Theta(u,\lambda)}\backslash W_\lambda$*, there exists a unique set of polynomials* $\{P^{u,\lambda}_{FG}\} \subset q\mathbb{Z}[q]$ *indexed by*

$$\left\{(F, G) \mid F, G \in \left(W_{\lambda,\Theta(u,\lambda)}\backslash W_\lambda\right)_{\leqslant_{u,\lambda} E}; G <_{u,\lambda} F\right\}$$

such that the function

$$\psi_{u,\lambda} : \left(W_{\lambda,\Theta(u,\lambda)}\backslash W_\lambda\right)_{\leqslant_{u,\lambda} E} \longrightarrow \mathcal{H}_{\Theta(u,\lambda)}, \quad F \mapsto \delta_F + \sum_{G<_{u,\lambda} F} P^{u,\lambda}_{FG}\delta_G$$

satisfies the following property: for any $F \in W_{\lambda,\Theta(u,\lambda)}\backslash W_\lambda$ *with* $F \neq W_{\lambda,\Theta(u,\lambda)}$*, there exist* $\alpha \in \Pi_\lambda$ *and* $c_G \in \mathbb{Z}$ *such that* $Fs_\alpha <_{u,\lambda} F$ *and*

$$T^{u,\lambda}_\alpha(\psi_{u,\lambda}(Fs_\alpha)) = \sum_{G\leqslant_{u,\lambda} F} c_G \psi_{u,\lambda}(G). \tag{4.1.3}$$

Moreover, the polynomials $P^{u,\lambda}_{FG}$ *do not depend on the choice of* E*, and they satisfy the parity condition*

$$P^{u,\lambda}_{CD} \in \mathbb{Z}[q^2, q^{-2}]q^{\ell_\lambda(C)-\ell_\lambda(D)}.$$

The elements $\psi_{u,\lambda}(C)$*'s will be referred to as (Whittaker) Kazhdan-Lusztig basis elements.*

Here $\ell_\lambda(C) = \ell_\lambda(w^{C|_\lambda})$, the length of the longest element in $C|_\lambda$.

We will write $P^{u,\lambda}_{CD}$ instead of $P^{u,\lambda}_{C|_\lambda,D|_\lambda}$ for convenience. Set $P^{u,\lambda}_{EE} = 1$ for all $E \in W_{\lambda,\Theta(u,\lambda)}\backslash W_\lambda$.

[1] Romanov actually denotes the map by φ. We reserve the notation φ to be used in the main algorithm 4.2.2.

4.2 Main algorithm

Recall that the category $\mathrm{Mod}_{coh}(\mathcal{D}_{C(w^D)}, N, \eta)$ is semisimple. Therefore, any complex $\mathcal{V}^\bullet$ of modules in this category is a direct sum of $\mathcal{O}^\eta_{C(w^D)}$'s at various degrees. We write $\chi_q \mathcal{V}^\bullet$ for its generating function (or q-Euler characteristic), i.e.

$$\chi_q \mathcal{V}^\bullet = \sum_{p\in\mathbb{Z}} \big(\operatorname{rank} H^p \mathcal{V}^\bullet\big) q^p.$$

Define the comparison map

$$\nu : \mathrm{Obj}\,\mathrm{Mod}_{coh}(\mathcal{D}_\lambda, N, \eta) \longrightarrow \mathcal{H}_\Theta,$$

$$\nu(\mathcal{F}) = \sum_{D\in W_\Theta\backslash W} \big(\chi_q i^!_{w^D}\mathcal{F}\big)\delta_D = \sum_{D\in W_\Theta\backslash W}\sum_{p\in\mathbb{Z}} \big(\operatorname{rank} H^p i^!_{w^D}\mathcal{F}\big) q^p \delta_D.$$

Here $i_{w^D} : C(w^D) \to X$ is the inclusion map. Clearly, this map can be extended to suitable derived categories.

The following easy property of ν is immediate:

Lemma 4.2.1.

$$\nu(\mathcal{I}(w^C, \lambda, \eta)) = \delta_C.$$

Proof. Let $D \in W_\Theta\backslash W$. Then $i^!_{w^D}\mathcal{I}(w^C,\lambda,\eta) = i^!_{w^D} i_{w^C+}\mathcal{O}^\eta_{C(w^C)}$. If $C = D$, this is $\mathcal{O}^\eta_{C(w^C)}$ by Kashiwara's theorem. Otherwise, this is 0 by base change. Hence the claim follows by the definition of ν. □

Theorem 4.2.2 (Kazhdan-Lusztig Algorithm for Whittaker modules)**.** *Fix a character $\eta : \mathfrak{n} \to \mathbb{C}$. For any $\lambda \in \mathfrak{h}^*$, there exists a unique map*

$$\varphi_\lambda : W_\Theta\backslash W \longrightarrow \mathcal{H}_\Theta$$

such that for any $C \in W_\Theta\backslash W$, if we write u for the unique element in $A_{\Theta,\lambda}$ such that C is contained in $W_\Theta u W_\lambda$, the following conditions hold:

(1) for some $P^{u,\lambda}_{CD} \in q\mathbb{Z}[q]$,

$$\varphi_\lambda(C) = \delta_C + \sum_{\substack{D\in W_\Theta\backslash W\\ D<_{u,\lambda} C}} P^{u,\lambda}_{CD}\delta_D.$$

(2) for any $\alpha \in \Pi\cap\Pi_\lambda$ with $Cs_\alpha < C$, there exist $c_D \in \mathbb{Z}$ such that

$$T_\alpha(\varphi_\lambda(Cs_\alpha)) = \sum_{\substack{D\in W_\Theta\backslash W\\ D\leqslant_{u,\lambda} C}} c_D\,\varphi_\lambda(D)$$

(3) *for any* $\beta \in \Pi - \Pi_\lambda$ *such that* $Cs_\beta < C$,

$$\varphi_{s_\beta \lambda}(Cs_\beta) = \varphi_\lambda(C)s_\beta$$

(recall that the right action $\mathcal{H}_\Theta \circlearrowleft W$ *is given by* $\delta_C \cdot w = \delta_{Cw}$*).*

(4) *The polynomials* $P_{CD}^{u,\lambda}$ *are Whittaker Kazhdan-Lusztig polynomials for* $(W_\lambda, \Pi_\lambda, \Theta(u, \lambda))$ *defined in* 4.1.2.

Moreover, the map φ_λ *is given by*

$$\varphi_\lambda(C) = \nu(\mathcal{L}(w^C, \lambda, \eta)).$$

If λ is integral, this reduces to the main theorem of Romanov [Rom21, Theorem 11].

A few remarks are in order.

First, if we ignore the last part of the theorem (that $\varphi_\lambda(C) = \nu(\mathcal{L}(w^C, \lambda, \eta))$), then the theorem becomes completely combinatorial. The main content of the theorem is that φ_λ is given by the comparison map ν. In other words, this theorem says that the relations between standard and irreducible modules in the category $\mathrm{Mod}_{coh}(\mathcal{D}_\lambda, N, \eta)$ are captured by various $\mathcal{H}_{\Theta(u,\lambda)}$'s. More precisely, in view of the geometric picture §1.4, the theorem says that the composition

$$\mathrm{Obj}\,\mathrm{Mod}_{coh}(\mathcal{D}_\lambda, N, \eta) \xrightarrow{\nu} \mathcal{H}_\Theta \xrightarrow{(-)|_\lambda} \bigoplus_{u \in A_{\Theta,\lambda}} \mathcal{H}_{\Theta(u,\lambda)}$$

sends irreducible modules to Whittaker Kazhdan-Lusztig basis elements and standard modules to the standard basis. Therefore, when specialized to $q = -1$ and passed to the Grothendieck group, the coefficient of a standard module in an irreducible module is given by Whittaker Kazhdan-Lusztig polynomials. Details can be found in Chapter 5.

Second, parts (1) through (3) of the theorem provides an algorithm for computing the coefficients $P_{CD}^{u,\lambda}$'s without referring to their original definition as Whittaker Kazhdan-Lusztig polynomials. We will demonstrate how to run this algorithm in §A.1.

Third, all parts of the theorem have simple geometric intuitions. (1) comes from the fact that $\mathcal{L}(w^C, \lambda, \eta)$ is supported on the closure of $C(w^C)$, so the pullback of $\mathcal{L}(w^C, \lambda, \eta)$ to a cell is nonzero only if that cell is on the closure of $C(w^C)$. Of course, (1) says more than this: $i^!_{w^D}\mathcal{L}(w^C, \lambda, \eta)$ can still be zero if $C(w^D) \subset \overline{C(w^C)}$. This happens if the modules on $C(w^C)$ and $C(w^D)$ are not in the same block. (2) reflects the action of the U-functor on irreducible Whittaker modules. See the comment after 3.1.8. (3) reflects the fact that non-integral intertwining functor is an equivalence of categories. So information of irreducibles in one category is fully translated to another category. Because of the usage of these intertwining functors, λ will be mapped to different chambers. Therefore it is necessary that the theorem is stated in a way that works for any λ. Once the theorem is established, one can choose λ to be antidominant so that localization can be used. For (4), the reader should refer to §2.5 for the idea behind the proof.

Let us begin the proof of the theorem. As in Chapter 3, we omit writing η in the proofs. Uniqueness is determined by (1), (4), and the uniqueness of Whittaker Kazhdan-Lusztig polynomials. For existence, we will show that $\varphi_\lambda(C) = \nu(\mathcal{L}(w^C, \lambda))$ satisfies the requirements (1)-(4) by induction on $\ell(w^C)$.

Consider the base case $\ell(w^C) = \ell(w_\Theta)$, that is, $C = W_\Theta$, $w^C = w_\Theta$. The argument for this case in the same as in [Rom21]. We include the details because it is short. Any composition factor of the standard module $\mathcal{I}(w_\Theta, \lambda)$ is supported on cells $C(w)$ in the closure of $C(w_\Theta)$. But any such w are in W_Θ with $w \leqslant w_\Theta$. In particular, w is not the longest element in its right W_Θ-coset unless $w = w_\Theta$. So there is no module supported on $C(w)$ unless $w = w_\Theta$. Hence the only composition factors are supported on $C(w_\Theta)$. By pulling back to $C(w_\Theta)$, we see that there is only one such factor, namely $\mathcal{L}(w_\Theta, \lambda)$. Thus $\mathcal{I}(w_\Theta, \lambda) = \mathcal{L}(w_\Theta, \lambda)$. As a result

$$\nu(\mathcal{L}(w_\Theta, \lambda)) = \nu(\mathcal{I}(w_\Theta, \lambda)) = \delta_{W_\Theta}$$

by 4.2.1. Therefore, the function $\varphi_\lambda(C)$ satisfies (1) for $C = W_\Theta$. The conditions (2)-(4) are void. This completes the base case.

Now consider the case $\ell(w^C) = k > \ell(w_\Theta)$. The verification of (1)-(4) for C is divided into subsections.

4.3 Proof of 4.2.2(3)

Assume $\beta \in \Pi - \Pi_\lambda$ is such that $Cs_\beta < C$. By definition,

$$\begin{aligned}
\varphi_\lambda(C)s_\beta &= \Big(\sum_{D \in W_\Theta \backslash W} \big(\chi_q i^!_{w^D} \mathcal{L}(w^C, \lambda)\big)\delta_D \Big) s_\beta \\
&= \sum_{D \in W_\Theta \backslash W} \big(\chi_q i^!_{w^D} \mathcal{L}(w^C, \lambda)\big)\delta_{Ds_\beta}
\end{aligned}$$

and

$$\begin{aligned}
\varphi_{s_\beta\lambda}(Cs_\beta) &= \sum_{D \in W_\Theta \backslash W} \big(\chi_q i^!_{w^D} \mathcal{L}(w^{Cs_\beta}, s_\beta\lambda)\big)\delta_D \\
&= \sum_{D \in W_\Theta \backslash W} \big(\chi_q i^!_{w^{Ds_\beta}} \mathcal{L}(w^{Cs_\beta}, s_\beta\lambda)\big)\delta_{Ds_\beta}
\end{aligned}$$

where in the last equality we rearranged the sum by the bijection $W_\Theta \backslash W \xrightarrow{\sim} W_\Theta \backslash W$, $D \mapsto Ds_\beta$. Hence it suffices to show that

$$\chi_q i^!_{w^D} \mathcal{L}(w^C, \lambda) = \chi_q i^!_{w^{Ds_\beta}} \mathcal{L}(w^{Cs_\beta}, s_\beta\lambda)$$

for any $D \in W_\Theta \backslash W$, which amounts to

$$\operatorname{rank} H^p i^!_{w^D} \mathcal{L}(w^C, \lambda) = \operatorname{rank} H^p i^!_{w^{D s_\beta}} \mathcal{L}(w^{C s_\beta}, s_\beta \lambda)$$

for any $p \in \mathbb{Z}$. This follows by 3.2.12.

4.4 Proof of 4.2.2(2)

Writing $\varphi_\lambda(C) = \nu(\mathcal{L}(w^C, \lambda))$, 4.2.2(2) reads

$$(T_\alpha \circ \nu)(\mathcal{L}(w^C s_\alpha, \lambda)) = \sum_{D \leqslant_{u,\lambda} C} c_D \nu(\mathcal{L}(w^D, \lambda)), \tag{4.2.2(2)}$$

which resembles

$$U_\alpha \mathcal{L}(w^C s_\alpha, \lambda) = \bigoplus_{D \leqslant C} \mathcal{L}(w^D, \lambda)^{\oplus c_D}$$

from Theorem 3.1.8. We are going to show that

$$(T_\alpha \circ \nu)(\mathcal{L}(w^C s_\alpha, \lambda)) = (\nu \circ U_\alpha)(\mathcal{L}(w^C s_\alpha, \lambda)). \tag{4.4.1}$$

Showing this will take up most of the work. Combined with the above equation for U_α, it leads to

$$(T_\alpha \circ \nu)(\mathcal{L}(w^C s_\alpha, \lambda)) = \sum_{D \leqslant C} c_D \nu(\mathcal{L}(w^D, \lambda))$$

after applying ν. This is close to what we wanted. Of course, 4.2.2(2) has fewer terms on the right hand side, but this will automatically follow from the proof of (4.4.1). This part of the argument is very similar to the ones in [Rom21] and [Mil]. The only modification is a little extra care in order to obtain the restricted sum on the right side of 4.2.2(2). Familiar readers can skip to the next section.

Let us start proving (4.4.1). For simplicity, we write $\mathcal{L} = \mathcal{L}(w^C s_\alpha, \lambda)$. By induction assumption, 4.2.2(1)(4) applies to $\mathcal{L}$, which reads

$$\varphi_\lambda(C s_\alpha) = \nu(\mathcal{L}) = \delta_{C s_\alpha} + \sum_{D <_{u,\lambda} C s_\alpha} P^{u,\lambda}_{C s_\alpha, D} \delta_D.$$

Compared with the definition of $\nu(\mathcal{L})$, we see that $P^{u,\lambda}_{CD} = \chi_q i^!_{w^D} \mathcal{L}$ whenever $D \leqslant_{u,\lambda} C$, and $0 = \chi_q i^!_{w^D} \mathcal{L}$ whenever $D \nleqslant_{u,\lambda} C$. We record this as a separate lemma for later use.

Lemma 4.4.2. *Suppose 4.2.2(1) holds for $C s_\alpha$. Then*

$$\chi_q i^!_{w^D} \mathcal{L}(w^C s_\alpha, \lambda, \eta) = \begin{cases} P^{u,\lambda}_{C s_\alpha, D} & D \leqslant_{u,\lambda} C \\ 0 & D \nleqslant_{u,\lambda} C. \end{cases}$$

Therefore, the left side of the desired equation can be rewritten as

$$T_\alpha(\nu(\mathcal{L})) = T_\alpha \delta_{C s_\alpha} + \sum_{D <_{u,\lambda} C s_\alpha} (\chi_q i^!_{w^D} \mathcal{L}) T_\alpha \delta_D.$$

We regroup the sum as

$$T_\alpha(\nu(\mathcal{L})) = \sum_{\substack{D s_\alpha < D \\ D \leqslant_{u,\lambda} C}} \Big((\chi_q i^!_{w^D s_\alpha} \mathcal{L}) T_\alpha \delta_{D s_\alpha} + (\chi_q i^!_{w^D} \mathcal{L}) T_\alpha \delta_D \Big).$$

The right side of the desired equation can also be rewritten:

$$\begin{aligned} \nu(U_\alpha \mathcal{L}) = & \sum_{D s_\alpha = D} (\chi_q i^!_{w^D} U_\alpha \mathcal{L}) \delta_D \\ & + \Big(\sum_{\substack{D s_\alpha < D \\ D \nleqslant_{u,\lambda} C}} + \sum_{\substack{D s_\alpha < D \\ D \leqslant_{u,\lambda} C}} \Big) \Big((\chi_q i^!_{w^D s_\alpha} U_\alpha \mathcal{L}) \delta_{D s_\alpha} + (\chi_q i^!_{w^D} U_\alpha \mathcal{L}) \delta_D \Big). \end{aligned} \tag{4.4.3}$$

To show that this is the same as $T_\alpha(\nu(\mathcal{L}))$, it is enough to show that the first two sums are zero, and that

$$(\chi_q i^!_{w^D s_\alpha} \mathcal{L}) T_\alpha \delta_{D s_\alpha} + (\chi_q i^!_{w^D} \mathcal{L}) T_\alpha \delta_D = (\chi_q i^!_{w^D s_\alpha} U_\alpha \mathcal{L}) \delta_{D s_\alpha} + (\chi_q i^!_{w^D} U_\alpha \mathcal{L}) \delta_D \tag{4.4.4}$$

for those D's in the third sum. To achieve this, we need to relate $i^! U_\alpha \mathcal{L}$ with $i^! \mathcal{L}$.

Let $D \in W_\Theta \backslash W$ be arbitrary for now. Before pulling back to $C(w^D)$, we want to first pull back to $p_\alpha^{-1}(p_\alpha(C(w^D)))$. Write $O = p_\alpha(C(w^D))$ and $X_O = p_\alpha^{-1}(O)$. We then have the following diagram

$$\begin{array}{ccc} X_O & \xrightarrow{\ s\ } & X \\ {\scriptstyle \pi_\alpha}\big\downarrow & & \big\downarrow{\scriptstyle p_\alpha} \\ O & \longrightarrow & X_\alpha \end{array} .$$

By base change,

$$s^! U_\alpha \mathcal{L}[1] = \pi^!_\alpha \pi_{\alpha +} s^! \mathcal{L}. \tag{4.4.5}$$

Suppose $D s_\alpha = D$, then $w^D s_\alpha < w^D$ (because w^D is the longest element in D), so π_α restricts to an N-equivariant isomorphism $\underset{\sim}{i} : C(w^D s_\alpha) \xrightarrow{\sim} O$. Because $w^D s_\alpha$ is not the longest element in D, there is no nontrivial η-twisted sheaf on $C(w^D s_\alpha)$, and the same is true for O. Hence $\pi_{\alpha+} s^! \mathcal{L} = 0$, and $s^! U_\alpha \mathcal{L} = 0$. Further pulling back to $C(w^D)$ from X_O, we see that $i^!_{w^D} U_\alpha \mathcal{L} = 0$. As a result, the first sum in (4.4.3) vanishes.

Suppose $D s_\alpha < D$. X_O is the inclusion of two cells, which form the following diagram

$$\begin{array}{ccccc} C(w^D) & \xrightarrow{\ j\ } & X_O & \xleftarrow{\ i\ } & C(w^D s_\alpha) \\ & {\scriptstyle q_\alpha}\searrow & {\scriptstyle \pi_\alpha}\big\downarrow & \cong\ \swarrow{\scriptstyle \underset{\sim}{i}} & \\ & & O & & \end{array} .$$

where i is closed and j is open. We will write the pullbacks of $U_\alpha\mathcal{L}$ to the cells in terms of the pullbacks of $\mathcal{L}$. This is achieved by examining the distinguished triangle for the immersions i and j, post composed with $\pi_\alpha^!\pi_{\alpha+}$:

$$\begin{array}{ccc} \pi_\alpha^!\pi_{\alpha+}i_+i^! & \longrightarrow & \pi_\alpha^!\pi_{\alpha+} \\ {\scriptstyle [1]}\nwarrow & & \swarrow \\ & \pi_\alpha^!\pi_{\alpha+}j_+j^! & \end{array} .$$

On the top-left corner, $\pi_{\alpha+}i_+ = \underset{\sim}{i}_+$; on the bottom vertex, $\pi_{\alpha+}j_+ = q_{\alpha+}$. So

$$\begin{array}{ccc} \pi_\alpha^!\underset{\sim}{i}_+i^! & \longrightarrow & \pi_\alpha^!\pi_{\alpha+} \\ {\scriptstyle [1]}\nwarrow & & \swarrow \\ & \pi_\alpha^!q_{\alpha+}j^! & \end{array} . \tag{4.4.6}$$

Further applying $i^!$, then $i^!\pi_\alpha^!$ is isomorphic to the identity map on $C(w^D s_\alpha)$. So we get

$$\begin{array}{ccc} i^! & \longrightarrow & i^!\pi_\alpha^!\pi_{\alpha+} \\ {\scriptstyle [1]}\nwarrow & & \swarrow \\ & q_{\alpha+}j^! & \end{array} .$$

Its long exact sequence on cohomologies reads

$$\cdots \longrightarrow H^p i^! \longrightarrow H^p i^!\pi_\alpha^!\pi_{\alpha+} \longrightarrow H^p q_{\alpha+}j^! \longrightarrow \cdots .$$

Note that $q_\alpha : C(w^D) \to O$ is isomorphic to a coordinate projection between affine spaces of relative dimension 1 (see the proof of 3.2.12). So $q_{\alpha+}$ sends $\mathcal{O}^\eta_{C(w^D)}$ to $\mathcal{O}^\eta_{C(w^D s_\alpha)}[1]$. The same can be said for $q_{\alpha+}j^!$, namely

$$\operatorname{rank} H^p q_{\alpha+}j^! = \operatorname{rank} H^{p+1}j^!$$

on η-twisted sheaves since their images under $j^!$ are direct sums of $\mathcal{O}^\eta_{C(w^D)}$'s at various degrees. So (by slight abuse of notation) the above long exact sequence becomes

$$\cdots \longrightarrow H^p i^! \longrightarrow H^p i^!\pi_\alpha^!\pi_{\alpha+} \longrightarrow {\mathcal{O}^\eta_{C(w^D s_\alpha)}}^{\oplus \operatorname{rank} H^{p+1}j^!} \longrightarrow \cdots . \tag{4.4.7}$$

If we instead apply $j^!$ to (4.4.6), we get

$$\begin{array}{ccc} q_\alpha^! i^! & \longrightarrow & i^!\pi_\alpha^!\pi_{\alpha+} \\ {\scriptstyle [1]}\nwarrow & & \swarrow \\ & q_\alpha^! q_{\alpha+}j^! & \end{array}$$

where we have used the fact that $q_\alpha^! = j^!\pi_\alpha^!$ to rewrite some terms. We again take the long exact sequence on cohomologies. Using the property of q_α discussed above, $q_\alpha^! = q_\alpha^+[1]$, and $q_\alpha^! q_{\alpha+} = [2]$ on η-twisted sheaves on $C(w^D)$. So the sequence becomes

$$\cdots \longrightarrow {\mathcal{O}^\eta_{C(w^D)}}^{\oplus \operatorname{rank} H^{p+1}i^!} \longrightarrow H^p j^!\pi_\alpha^!\pi_{\alpha+} \longrightarrow H^{p+2}j^! \longrightarrow \cdots . \tag{4.4.8}$$

Now we apply these two long exact sequences to $s^!\mathcal{L}$. Then $i^!\pi_\alpha^!\pi_{\alpha+}s^!\mathcal{L} = i^!_{w^D s_\alpha} U_\alpha\mathcal{L}[1]$ (see (4.4.5)), and the same thing but for $j^!$ is equal to $i^!_{w^D} U_\alpha\mathcal{L}[1]$. Suppose D is in the second sum of (4.4.3), i.e. suppose $D \not\leqslant_{u,\lambda} C$. Then by lifting property of Bruhat order [BB05, 2.2.7], we see that $D, Ds_\alpha \not\leqslant_{u,\lambda} C, Cs_\alpha$. Hence $i^!_{w^D}\mathcal{L} = i^!_{w^D s_\alpha}\mathcal{L} = 0$ by Lemma 4.4.2. As a result, the long exact sequences force the vanishing of the pullback of $U_\alpha\mathcal{L}$ to the two cells, and the second sum in (4.4.3) vanishes.

It remains to examine the case $D \leqslant_{u,\lambda} C$. In this case $i^!_{w^D}\mathcal{L}$ is governed by $P^{u,\lambda}_{CD}$ (Lemma 4.4.2), which satisfies parity condition (see Definition 4.1.2):

$$H^p i^!_{w^D}\mathcal{L} = 0 \text{ whenever } p \not\equiv \ell_\lambda(Cs_\alpha) - \ell_\lambda(D).$$

As a result, the long exact sequences (4.4.7) (4.4.8) alternate between three consecutive vanishing terms and three possibly non-vanishing terms forming a short exact sequence. Taking ranks of these three-term sequences, we obtain

$$\begin{aligned}
\operatorname{rank} H^p i^!_{w^D s_\alpha} U_\alpha\mathcal{L} &= \operatorname{rank} H^{p-1} i^!_{w^D s_\alpha}\mathcal{L} + \operatorname{rank} H^p i^!_{w^D}\mathcal{L} \\
\operatorname{rank} H^p i^!_{w^D} U_\alpha\mathcal{L} &= \operatorname{rank} H^p i^!_{w^D s_\alpha}\mathcal{L} + \operatorname{rank} H^{p+1} i^!_{w^D}\mathcal{L}.
\end{aligned}$$

Hence the right side of (4.4.4) is

$$\begin{aligned}
&(\chi_q i^!_{w^D s_\alpha} U_\alpha\mathcal{L})\delta_{Ds_\alpha} + (\chi_q i^!_{w^D} U_\alpha\mathcal{L})\delta_D \\
=&\sum_p \big(\operatorname{rank} H^p i^!_{w^D s_\alpha} U_\alpha\mathcal{L}\big) q^p \delta_{Ds_\alpha} \\
&+\sum_p \big(\operatorname{rank} H^p i^!_{w^D} U_\alpha\mathcal{L}\big) q^p \delta_D \\
=&\sum_p \big(\operatorname{rank} H^{p-1} i^!_{w^D s_\alpha}\mathcal{L} + \operatorname{rank} H^p i^!_{w^D}\mathcal{L}\big) q^p \delta_{Ds_\alpha} \\
&+\sum_p \big(\operatorname{rank} H^p i^!_{w^D s_\alpha}\mathcal{L} + \operatorname{rank} H^{p+1} i^!_{w^D}\mathcal{L}\big) q^p \delta_D.
\end{aligned}$$

Here the first equality is by definition of χ_q, and in the last equation we have rewritten the ranks of pullbacks of $U_\alpha\mathcal{L}$ in terms of those of $\mathcal{L}$. The left side of (4.4.4) is

$$\begin{aligned}
&(\chi_q i^!_{w^D s_\alpha}\mathcal{L})T_\alpha\delta_{Ds_\alpha} + (\chi_q i^!_{w^D}\mathcal{L})T_\alpha\delta_D \\
=&\sum_p \Big(\big(\operatorname{rank} H^p i^!_{w^D s_\alpha}\mathcal{L}\big)\big(q^{p+1}\delta_{Ds_\alpha} q^p\delta_D\big) \\
&\qquad + \big(\operatorname{rank} H^p i^!_{w^D}\mathcal{L}\big)\big(q^p\delta_{Ds_\alpha} q^{p-1}\delta_D\big)\Big) \\
=&\sum_p \big(\operatorname{rank} H^{p-1} i^!_{w^D s_\alpha}\mathcal{L} + \operatorname{rank} H^p i^!_{w^D}\mathcal{L}\big) q^p \delta_{Ds_\alpha}
\end{aligned}$$

$$+\sum_{p}\left(\operatorname{rank} H^{p} i^{!}_{w^{D}s_{\alpha}}\mathcal{L}+\operatorname{rank} H^{p+1} i^{!}_{w^{D}}\mathcal{L}\right) q^{p}\delta_{D}.$$

Here the first equality is by definition of χ_q and T_α, and the second equation is obtained by rearranging the sum according to the basis elements $q^p\delta_{Ds_\alpha}$ and $q^p\delta_D$. Thus the two sides of (4.4.4) equal. Consequently $T_\alpha(\nu(\mathcal{L}))=\nu(U_\alpha\mathcal{L})$, and (4.4.1) holds.

In the course of this proof, we have seen that the first two sums in (4.4.3) vanish. This necessarily implies the same restriction for $U_\alpha\mathcal{L}$. Recall that $U_\alpha\mathcal{L}$ is a direct sum of $\mathcal{L}(w^D,\lambda)$'s (Theorem 3.1.8). If $\mathcal{L}(w^D,\lambda)$ appears in $U_\alpha\mathcal{L}$, then $i^!_{w^D}\mathcal{L}(w^D,\lambda)=\mathcal{O}^{\eta}_{C(w^D)}$ appears in $i^!_{w^D}U_\alpha\mathcal{L}$. This cannot happen if $D\not\leqslant_{u,\lambda}C$ because they contribute to the first two sums of (4.4.3). Hence such D's will not appear in $U_\alpha\mathcal{L}$. We record this as a corollary.

Corollary 4.4.9. *Suppose $\alpha\in\Pi\cap\Pi_\lambda$ and $Cs_\alpha<C$. There exist $c_D\in\mathbb{Z}$ depending on C and α so that*

$$U_\alpha\mathcal{L}(w^C s_\alpha,\lambda,\eta)=\bigoplus_{D\leqslant_{u,\lambda}C}\mathcal{L}(w^D,\lambda,\eta)^{\oplus c_D}.$$

In particular, $i^!_{w^D}U_\alpha\mathcal{L}(w^C s_\alpha,\lambda,\eta)=0$ whenever $D\not\leqslant_{u,\lambda}C$.

Thus

$$\begin{aligned}
T_\alpha(\varphi_\lambda(\mathcal{L}))&=T_\alpha(\nu(\mathcal{L}))=\nu(U_\alpha\mathcal{L})\\
&=\nu\Big(\bigoplus_{D\leqslant_{u,\lambda}C}\mathcal{L}(w^D,\lambda)^{\oplus c_D}\Big)\\
&=\sum_{D\leqslant_{u,\lambda}C}c_D\,\nu(\mathcal{L}(w^D,\lambda))\\
&=\sum_{D\leqslant_{u,\lambda}C}c_D\,\varphi_\lambda(D).
\end{aligned}$$

4.2.2(2) is now verified for C.

4.5 Proof of 4.2.2(1)

The idea is to find a simple reflection s so that $Cs<C$, and deduce information of C from that of Cs. If s is non-integral, we can use non-integral intertwining functor I_s to translate properties of Cs to C. If s is integral, we then use information about the U-functor.

Again we omit writing the η's.

Suppose there exists $\beta\in\Pi-\Pi_\lambda$ such that $Cs_\beta<C$. By induction hypothesis, 4.2.2(1) applies to Cs_β, which says

$$\varphi_{s_\beta\lambda}(Cs_\beta)=\delta_{Cs_\beta}+\sum_{D<_{r,s_\beta\lambda}Cs_\beta}Q_D\delta_D,$$

for some polynomials $Q_D \in q\mathbb{Z}[q]$, where r is the unique element in $A_{\Theta,s_\beta\lambda}$ such that Cs_β is contained in $W_\Theta r W_{s_\beta\lambda}$. Applying 4.2.2(3) for C,

$$\varphi_\lambda(C) = \varphi_{s_\beta\lambda}(Cs_\beta)s_\beta = \delta_C + \sum_{D <_{r,s_\beta\lambda} Cs_\beta} Q_D \delta_{Ds_\beta}.$$

We want to rewrite the subscript of the sum. By 2.4.6 and its corollary, there exists $w \in W_\Theta$ with $wr = us_\beta$. Hence

$$\begin{aligned} W_\Theta r W_{s_\beta\lambda} &= (W_\Theta w) r (s_\beta W_\lambda s_\beta) \\ &= W_\Theta (wr) s_\beta W_\lambda s_\beta \\ &= W_\Theta (us_\beta) s_\beta W_\lambda s_\beta \\ &= W_\Theta u W_\lambda s_\beta, \end{aligned}$$

and we see that $D \in W_\Theta \backslash W_\Theta r W_{s_\beta\lambda}$ if and only if $Ds_\beta \in W_\Theta \backslash W_\Theta u W_\lambda$. By 2.4.8,

$$D <_{r,s_\beta\lambda} Cs_\beta \iff Ds_\beta <_{u,\lambda} C.$$

Hence

$$\begin{aligned} \varphi_\lambda(C) &= \delta_C + \sum_{Ds_\beta <_{u,\lambda} C} Q_D \delta_{Ds_\beta} \\ &= \delta_C + \sum_{E <_{u,\lambda} C} Q_D \delta_E \end{aligned}$$

for some $Q_D \in q\mathbb{Z}[q]$, and 4.2.2(1) holds for C in this case.

If such β does not exist, then there exists a simple integral root α with $Cs_\alpha < C$. From Theorem 3.1.8, we know $\mathcal{L}(w^C, \lambda)$ is a direct summand of $U_\alpha \mathcal{L}(Cs_\alpha, \lambda)$. So the coefficients of the polynomial $\chi_q i^!_{w^D} \mathcal{L}(w^C, \lambda)$ (which are non-negative integers) must be dominated by those of $\chi_q i^!_{w^D} U_\alpha \mathcal{L}(w^C s_\alpha, \lambda)$. On the other hand, we know from Corollary 4.4.9 that the latter polynomial vanishes for $D \not\leqslant_{u,\lambda} C$. So the former also vanishes for those D's. Hence

$$\begin{aligned} \varphi_\lambda(C) &= \sum_{D \in W_\Theta \backslash W} \left(\chi_q i^!_{w^D} \mathcal{L}(w^C, \lambda)\right) \delta_D \\ &= \sum_{D \leqslant_{u,\lambda} C} \left(\chi_q i^!_{w^D} \mathcal{L}(w^C, \lambda)\right) \delta_D. \end{aligned}$$

It suffices to compute the remaining coefficients. The case $D = C$ is treated in 3.1.18: $i^!_{w^C} \mathcal{L}(w^C, \lambda, \eta) = \mathcal{O}^\eta_{C(w^C)}$. Hence the coefficient of δ_C is 1. For $D < C$, we know $H^0 i^!_{w^D}$ takes sections supported in $C(w^D)$. We also know that $\mathcal{L}(w^C, \lambda)$ has no section supported in $\partial C(w^C) \supset C(w^D)$. Hence $H^0 i^!_{w^D} \mathcal{L}(w^C, \lambda) = 0$ and the coefficient of δ_D has no constant term. Thus 4.2.2(1) holds for C.

4.6 Proof of 4.2.2(4)

Based on our definition of parabolic Kazhdan-Lusztig polynomials 4.1.2, we need to find $\alpha \in \Pi_\lambda$ such that $Cs_\alpha <_{u,\lambda} C$ and equation (4.1.3) holds for the function

$$\psi_{u,\lambda}(C|_\lambda) := \varphi_\lambda(C)|_\lambda.$$

See §2.5 for an explanation of the geometric idea behind this proof.

If α can be chosen to be in $\Pi \cap \Pi_\lambda$, then by the following lemma, (4.1.3) follows from 4.2.2(2) for C.

Lemma 4.6.1. *Let $\alpha \in \Pi \cap \Pi_\lambda$. Then for each $u \in A_{\Theta,\lambda}$*

$$(-)|_\lambda \circ T_\alpha = T_\alpha^{u,\lambda} \circ (-)|_\lambda$$

as maps from $\mathrm{ind}_\lambda \mathcal{H}_{\Theta(u,\lambda)} \subseteq \mathcal{H}_\Theta$ to $\mathcal{H}_{\Theta(u,\lambda)}$ (the maps ind_λ and $(-)|_\lambda$ are defined in §4.1). In other words, the following diagram commutes

$$\begin{array}{ccc} \mathcal{H}_\Theta & \xrightarrow{\quad T_\alpha \quad} & \mathcal{H}_\Theta \\ \big\downarrow {\scriptstyle (-)|_\lambda} & & \big\downarrow {\scriptstyle (-)|_\lambda} \\ \bigoplus_{u \in A_{\Theta,\lambda}} \mathcal{H}_{\Theta(u,\lambda)} & \xrightarrow{\bigoplus_u T_\alpha^{u,\lambda}} & \bigoplus_{u \in A_{\Theta,\lambda}} \mathcal{H}_{\Theta(u,\lambda)}. \end{array}$$

The proof is straightforward. It consists of unwrapping definitions and using the fact that ind_λ preserves partial orders on right cosets 2.4.3.

If such α cannot be found, we will need to use non-integral intertwining functors to move α to some simple root $s_{\beta_s} \cdots s_{\beta_1} \alpha = z^{-1}\alpha$ and move $\mathcal{L}(w^C, \lambda, \eta)$ to some irreducible module supported on a smaller orbit where 4.2.2(2) is known to hold, and then translate the induction assumption there back. The messiness of the argument below are merely the result of careful bookkeeping. The translation step requires the following lemma. The proof is similar to the previous one, using 2.4.8 instead of 2.4.3.

Lemma 4.6.2. *Let $\alpha \in \Pi \cap \Pi_\lambda$, $\beta \in \Pi - \Pi_\lambda$. For any $u \in A_{\Theta,\lambda}$, let $r \in A_{\Theta, s_\beta\lambda}$ be the unique element such that $W_\Theta u s_\beta W_{s_\beta\lambda} = W_\Theta r W_{s_\beta\lambda}$. Then*

$$(s_\beta(-)s_\beta) \circ T_\alpha^{u,\lambda} = T_{s_\beta\alpha}^{r,s_\beta\lambda} \circ (s_\beta(-)s_\beta)$$

as maps from $\mathcal{H}_{\Theta(u,\lambda)}$ to $\mathcal{H}_{\Theta(r,s_\beta\lambda)}$, where $s_\beta(-)s_\beta$ denotes conjugation by s_β. In other words, the *following diagram commutes*

$$\begin{array}{ccc} \mathcal{H}_{\Theta(u,\lambda)} & \xrightarrow{\quad T_\alpha^{u,\lambda}\quad} & \mathcal{H}_{\Theta(u,\lambda)} \\ {\scriptstyle s_\beta(-)s_\beta}\big\downarrow & & \big\downarrow{\scriptstyle s_\beta(-)s_\beta} \\ \mathcal{H}_{\Theta(r,s_\beta\lambda)} & \xrightarrow{\quad T_{s_\beta\alpha}^{r,s_\beta\lambda}\quad} & \mathcal{H}_{\Theta(r,s_\beta\lambda)}. \end{array}$$

Choose $\alpha \in \Pi_\lambda$, $s \geqslant 0$ and $\beta_1, \dots, \beta_s \in \Pi$ such that if we write $z_0 = 1$, $z_i = s_{\beta_1} \cdots s_{\beta_i}$ and $z = z_s$, the following conditions hold:

(a) for any $0 \leqslant i \leqslant s-1$, β_{i+1} is non-integral to $z_i^{-1}\lambda$;

(b) $z^{-1}\alpha \in \Pi \cap \Pi_{z^{-1}\lambda}$;

(c) $Cs_\alpha <_{u,\lambda} C$;

(d) if $s > 0$, $Cz < C$;

(e) $Cs_\alpha z = Czs_{z^{-1}\alpha} < Cz$.

Such a choice exists by 2.5.1. Combining the lemmas with the diagram (2.4.9), we obtain a commutative diagram

$$\begin{array}{ccccc} & & \mathcal{H}_\Theta & \xrightarrow{\qquad T_{z^{-1}\alpha}\qquad} & \mathcal{H}_\Theta \\ & \swarrow{\scriptstyle (-)z^{-1}} & \big\downarrow{\scriptstyle (-)|_{z^{-1}\lambda}} & & \big\downarrow{\scriptstyle (-)|_{z^{-1}\lambda}} \\ \mathcal{H}_\Theta & & \bigoplus_{r\in A_{\Theta,z^{-1}\lambda}} \mathcal{H}_{\Theta(r,z^{-1}\lambda)} & \xrightarrow{\oplus_r T^{r,z^{-1}\lambda}_{z^{-1}\alpha}} & \bigoplus_{r\in A_{\Theta,z^{-1}\lambda}} \mathcal{H}_{\Theta(r,z^{-1}\lambda)} \\ \big\downarrow{\scriptstyle (-)|_\lambda} & \swarrow{\scriptstyle z(-)z^{-1}} & & \swarrow{\scriptstyle z(-)z^{-1}} & \\ \bigoplus_{u\in A_{\Theta,\lambda}} \mathcal{H}_{\Theta(u,\lambda)} & \xrightarrow{\oplus_u T_\alpha^{u,\lambda}} & \bigoplus_{u\in A_{\Theta,\lambda}} \mathcal{H}_{\Theta(u,\lambda)} & & \end{array} \quad . \tag{4.6.3}$$

Since $Cz < C$, the induction assumption applies to Cz and $z^{-1}\lambda$. In particular, if we apply 4.2.2(2) to $Czs_{z^{-1}\alpha} < Cz$ and $z^{-1}\lambda$, we obtain the equation

$$T_{z^{-1}\alpha}\big(\varphi_{z^{-1}\lambda}(Czs_{z^{-1}\alpha})\big) = \sum_{D \leqslant_{r,z^{-1}\lambda} Cz} c_D \varphi_{z^{-1}\lambda}(D) \tag{4.6.4}$$

where r is the unique element in $A_{\Theta,z^{-1}\lambda}$ such that $Cz \in W_\Theta \backslash W_\Theta r W_{z^{-1}\lambda}$.

Claim 4.6.5. If we apply $z(-)z^{-1} \circ (-)|_{z^{-1}\lambda}$ to both sides, (4.6.4) becomes

$$T_\alpha^{u,\lambda}(\psi_{u,\lambda}(C|_\lambda)) = \sum_{E \leqslant_{u,\lambda} C} c_D \psi_{u,\lambda}(E|_\lambda).$$

Consequently, $\alpha \in \Pi_\lambda$ is such that $Cs_\alpha <_{u,\lambda} C$ and equation (4.1.3) holds for Cs_α. By 4.1.2, the polynomials $P^{u,\lambda}_{CD}$ are parabolic Kazhdan-Lusztig polynomials for $(W_\lambda, \Pi_\lambda, \Theta(u,\lambda))$. Thus 4.2.2(4) holds for C.

It remains to prove the claim. If we view $\varphi_{z^{-1}\lambda}(Czs_{z^{-1}\alpha})$ as an element in the middle-top $\mathcal{H}_\Theta$ in the diagram, then after applying $(-)|_{z^{-1}\lambda}$ and $z(-)z^{-1}$, the left side of (4.6.4) lands in $\mathcal{H}_{\Theta(u,\lambda)}$ at the bottom middle position of the diagram through the rightmost path. Going through the leftmost path instead, this element in $\mathcal{H}_{\Theta(u,\lambda)}$ becomes

$$T^{u,\lambda}_\alpha\big(\varphi_{z^{-1}\lambda}(Czs_{z^{-1}\alpha})z^{-1}|_\lambda\big).$$

Rewrite $Czs_{z^{-1}\alpha} = Cs_\alpha z$ and use 4.2.2(3) repeatedly for Cs_α, the above quantity becomes

$$T^{u,\lambda}_\alpha\big(\varphi_\lambda(Cs_\alpha)|_\lambda\big) = T^{u,\lambda}_\alpha(\psi_{u,\lambda}(C|_\lambda)).$$

Viewing the right side of (4.6.4) as an element in the middle-top $\mathcal{H}_\Theta$ in the diagram, $(-)|_{z^{-1}\lambda}$ and $z(-)z^{-1}$ sends it to $\mathcal{H}_{\Theta(u,\lambda)}$ at the bottom-left along the middle path. Going through the leftmost path instead, this element becomes

$$\sum_{D \leqslant_{r,z^{-1}\lambda} Cz} c_D\, \varphi_\lambda(Dz^{-1})|_\lambda = \sum_{D \leqslant_{r,z^{-1}\lambda} Cz} c_D\, \psi_{u,\lambda}((Dz^{-1})|_\lambda).$$

As in the first half of §4.5, we can rewrite the subscript of the sum. There is an element $w \in W_\Theta$ such that $wr = uz$ by 2.4.6. Hence

$$\begin{aligned} W_\Theta r W_{z^{-1}\lambda} &= W_\Theta w r z^{-1} W_\lambda z \\ &= W_\Theta u z z^{-1} W_\lambda z \\ &= W_\Theta u W_\lambda z, \end{aligned}$$

and $D \in W_\Theta \backslash W_\Theta r W_{z^{-1}\lambda}$ if and only if $Dz^{-1} \in W_\Theta \backslash W_\Theta u W_\lambda$. Moreover, by 2.4.8,

$$D \leqslant_{r,z^{-1}\lambda} Cz \iff Dz^{-1} \leqslant_{u,\lambda} C.$$

Hence the right side of (4.6.4) becomes

$$\sum_{Dz^{-1} \leqslant_{u,\lambda} C} c_D \psi_{u,\lambda}((Dz^{-1})|_\lambda) = \sum_{E \leqslant_{u,\lambda} C} c_D \psi_{u,\lambda}(E|_\lambda).$$

This proves the claim.

The proof of 4.2.2 is now complete.

Chapter 5

Character formula for irreducible modules

5.1 Regular case

By standard arguments, the algorithm 4.2.2 leads to a character formula for irreducible Whittaker modules with regular infinitesimal characters.

Let $\lambda \in \mathfrak{h}^*$ be antidominant regular. As explained in §1.3, Beilinson-Bernstein's localization and holonomic duality are equivalences of categories which send Whittaker modules to η-twisted $\mathcal{D}$-modules. Combined with the maps ν and $(-)|_{-\lambda}$, we obtain the composition

$$\mathcal{N}_{\theta,\eta} \xleftarrow{\Gamma(X,-)} \mathrm{Mod}_{coh}(\mathcal{D}_\lambda, N, \eta) \xrightarrow{\mathbb{D}} \mathrm{Mod}_{coh}(\mathcal{D}_{-\lambda}, N, \eta) \xrightarrow{\nu} \mathcal{H}_\Theta \xrightarrow{(-)|_{-\lambda}} \bigoplus_{u \in A_{\Theta,-\lambda}} \mathcal{H}_{\Theta(u,-\lambda)}, \tag{5.1.1}$$

under which

$$\begin{array}{ccccccccc} L(w^C\lambda,\eta) & \leftarrow\!\shortmid & \mathcal{L}(w^C,\lambda,\eta) & \mapsto & \mathcal{L}(w^C,-\lambda,\eta) & \mapsto & \varphi_{-\lambda}(C) & \mapsto & \varphi_{-\lambda}(C)|_{-\lambda}, \\ M(w^C\lambda,\eta) & \leftarrow\!\shortmid & \mathcal{M}(w^C,\lambda,\eta) & \mapsto & \mathcal{I}(w^C,-\lambda,\eta) & \mapsto & \delta_C & \mapsto & \delta_{C|_{-\lambda}}. \end{array}$$

Since $\chi_q|_{q=-1}$ is the usual Euler characteristic, the coefficients $\chi_q i^!_{w^D}\mathcal{F}$ in the definition of ν are additive with respect to short exact sequences. So ν factors through the Grothendieck group

$$\nu|_{q=-1} : K\,\mathrm{Mod}_{coh}(\mathcal{D}_\lambda, N, \eta) \longrightarrow \mathcal{H}_\Theta|_{q=-1}$$

which is an isomorphism by 4.2.1. Therefore we have an isomorphism of abelian groups

$$\begin{array}{ccc} K\mathcal{N}_{\theta,\eta} & \xrightarrow{\cong} & \displaystyle\bigoplus_{u \in A_{\Theta,-\lambda}} \mathcal{H}_{\Theta(u,-\lambda)}|_{q=-1} \\ [L(w^C\lambda,\eta)] & \mapsto & \varphi_\lambda(C)|_{-\lambda}|_{q=-1} \\ [M(w^C\lambda,\eta)] & \mapsto & \delta_{C|_{-\lambda}}|_{q=-1}. \end{array}$$

Hence 4.2.2(1) and (4) imply

$$[L(w^C\lambda,\eta)] = \sum_{D \leqslant_{u,-\lambda} C} P^{u,-\lambda}_{CD}(-1)[M(w^D\lambda,\eta)]$$

in $K\mathcal{N}_{\theta,\eta}$. Note that $\Sigma_\lambda = \Sigma_{-\lambda}$ as subsets of Σ and $W_\lambda = W_{-\lambda}$ as subgroups of W. Hence all the combinatorial structures defined based on λ and $-\lambda$ are canonically identified. In particular, $D \leqslant_{u,-\lambda} C$ if and only if $D \leqslant_{u,\lambda} C$, and $P^{u,-\lambda}_{CD} = P^{u,\lambda}_{CD}$. Further applying the character map, we thus obtain

Theorem 5.1.2 (Character formula: regular case)**.** *Let $\lambda \in \mathfrak{h}^*$ be antidominant and regular. Let $\eta : \mathfrak{n} \to \mathbb{C}$ be any character. For any $C \in W_\Theta\backslash W$, let $u \in A_{\Theta,\lambda}$ be the unique element such that $C \subseteq W_\Theta u W_\lambda$. Then*

$$\operatorname{ch} L(w^C\lambda,\eta) = \sum_{\substack{D\in W_\Theta\backslash W \\ D\leqslant_{u,\lambda} C}} P^{u,\lambda}_{CD}(-1) \operatorname{ch} M(w^D\lambda,\eta), \tag{5.1.3}$$

where the $P^{u,\lambda}_{CD}$'s are Whittaker Kazhdan-Lusztig polynomials for $(W_\lambda, \Pi_\lambda, \Theta(u,\lambda))$ as defined in 4.1.2.

When λ is integral, we have a simpler description, which we state separately.

Corollary 5.1.4 (Character formula: regular integral case)**.** *Let $\lambda \in \mathfrak{h}^*$ be antidominant, regular, and integral. Let $\eta : \mathfrak{n} \to \mathbb{C}$ be any character. For any $C \in W_\Theta\backslash W$,*

$$\operatorname{ch} L(w^C\lambda,\eta) = \sum_{\substack{D\in W_\Theta\backslash W \\ D\leqslant C}} P_{CD}(-1) \operatorname{ch} M(w^D\lambda,\eta),$$

where the P_{CD}'s are Whittaker Kazhdan-Lusztig polynomials for (W, Π, Θ) as defined in 4.1.1.

Inverting the matrix $(P_{CD}(-1))_{C,D}$, we recover the description in [MS97] and [Rom21] of multiplicities of irreducible Whittaker modules in standard Whittaker modules with antidominant regular integral infinitesimal characters.

At another extreme, when $\eta = 0$ (i.e. $\Theta = \varnothing$), we recover the well-known non-integral Kazhdan-Lusztig conjecture for category $\mathcal{O}$.

Corollary 5.1.5 (Kazhdan-Lusztig conjecture for category $\mathcal{O}$)**.** *Let $\lambda \in \mathfrak{h}^*$ be antidominant and regular. For any $w \in W$,*

$$\operatorname{ch} L(w\lambda) = \sum_{\substack{v\in W \\ v\leqslant_\lambda w}} P^\lambda_{wv}(-1) \operatorname{ch} M(v\lambda),$$

where the P^λ_{wv}'s are Kazhdan-Lusztig polynomials for $(W_\lambda, \Pi_\lambda, \varnothing)$ as defined in 4.1.2, $M(v\lambda)$ is the Verma module of highest weight $v\lambda - \rho$, and $L(w\lambda)$ is the unique irreducible quotient of $M(w\lambda)$ (recall that ρ is the half sum of roots in Σ^+).

5.2 Singular case

The singular case can be deduced from the regular case easily.

Let $\lambda \in \mathfrak{h}^*$ be antidominant and singular. We still have the maps in (5.1.1), but the exact functor $\Gamma(X, -)$ is no longer an equivalence of categories and only descends to a surjection $K\,\mathrm{Mod}_{coh}(\mathcal{D}_\lambda, N, \eta) \twoheadrightarrow K\mathcal{N}_{\theta,\eta}$ on Grothendieck groups. However, the identification $\Gamma(X, \mathcal{M}(w^D, \lambda, \eta)) = M(w^D\lambda, \eta)$ still holds [Rom21, Theorem 9]. Therefore, the argument for regular case produces the equality

$$\operatorname{ch}\Gamma(X, \mathcal{L}(w^C, \lambda, \eta)) = \sum_{D \leqslant_{u,\lambda} C} P^{u,\lambda}_{CD}(-1)\operatorname{ch} M(w^D\lambda, \eta). \tag{5.2.1}$$

However, $\Gamma(X, \mathcal{L}(w^C, \lambda, \eta))$ may be zero, and the $M(w^D\lambda, \eta)$'s may coincide for different D's. It remains to describe which $M(w^D\lambda, \eta)$'s coincide and which $\Gamma(X, \mathcal{L}(w^C, \lambda, \eta))$'s are zero.

The first question has an easy answer. Recall that for $C, D \in W_\Theta\backslash W$, $M(w^D\lambda, \eta) = M(w^C\lambda, \eta)$ if and only if $W_\Theta w^D\lambda = W_\Theta w^C\lambda$ (§1.1). Let W^λ be the stabilizer of λ in W. Then the above condition is equivalent to $W_\Theta w^D W^\lambda = W_\Theta w^C W^\lambda$, i.e. that C and D are in the same (W_Θ, W^λ)-coset.

Lemma 5.2.2. *Let $\lambda \in \mathfrak{h}^*$ be antidominant and let $\eta : \mathfrak{n} \to \mathbb{C}$ be a character. The following are equivalent:*

(a) $M(w^C\lambda, \eta) = M(w^D\lambda, \eta)$;

(b) $\Gamma(X, \mathcal{M}(w^C, \lambda, \eta)) = \Gamma(X, \mathcal{M}(w^D, \lambda, \eta))$;

(c) C *and* D *are in the same double* (W_Θ, W^λ)*-coset.*

Therefore, for a fixed standard Whittaker module M, there is a unique double coset $W_\Theta v W^\lambda$ such that $\Gamma(X, \mathcal{M}(w^D, \lambda, \eta)) = M$ if and only if $D \in W_\Theta\backslash W_\Theta v W^\lambda$.

The following proposition answers the second question.

Proposition 5.2.3. *Let $\lambda \in \mathfrak{h}^*$ be antidominant and let $\eta : \mathfrak{n} \to \mathbb{C}$ be a character. Let $v \in W$. Then the set $W_\Theta\backslash W_\Theta v W^\lambda$ of right W_Θ-cosets contains a unique smallest element* C. *Furthermore,*

(a) $\Gamma(X, \mathcal{L}(w^C, \lambda, \eta)) = L(w^C\lambda, \eta) \neq 0$, *and*

(b) $\Gamma(X, \mathcal{L}(w^D, \lambda, \eta)) = 0$ *for any* $D \in W_\Theta\backslash W_\Theta v W^\lambda$ *not equal to* C.

In other words, for a fixed standard Whittaker module M, among all the costandard $\mathcal{D}_\lambda$-modules that realize M, the irreducible quotient of the one with the smallest support realizes the unique irreducible submodule of M.

Proof. Write $M = M(v\lambda, \eta)$ and $L = L(v\lambda, \eta)$.

First, there is one and at most one D in $W_\Theta\backslash W$ with $\Gamma(X, \mathcal{L}(w^D, \lambda, \eta)) = L$. This is because, by the theory of localization, there is a unique irreducible $\mathcal{D}_\lambda$-module $\mathcal{V}$ with $\Gamma(X, \mathcal{V}) = L$ (see [Mil, Chapter 3 §5 Proposition 5.2]; in fact, $\mathcal{V}$ is the unique irreducible quotient of $\mathcal{D}_\lambda \otimes_{\mathcal{U}_\theta} L$). By the classification of irreducible twisted Harish-Chandra sheaves, $\mathcal{V}$ equals to $\mathcal{L}(w^D, \lambda, \eta)$ for a single $D \in W_\Theta \backslash W$.

Since $\mathcal{L}(w^D, \lambda, \eta)$ is the unique irreducible quotient of $\mathcal{M}(w^D, \lambda, \eta)$ and $\Gamma(X, -)$ is exact on $\mathcal{D}_\lambda$-modules, $L = \Gamma(X, \mathcal{L}(w^D, \lambda, \eta))$ equals the unique irreducible quotient $L(w^D\lambda, \eta)$ of $M(w^D\lambda, \eta)$. This forces $M = M(w^D\lambda, \eta)$. Hence, by the preceding lemma, D is contained in the double coset $W_\Theta v W^\lambda$.

It remains to show that such a D is minimum in $W_\Theta \backslash W_\Theta v W^\lambda$. Let C be a minimal element in $W_\Theta \backslash W_\Theta v W^\lambda$. The composition factors of $\mathcal{M}(w^C, \lambda, \eta)$ consist of certain $\mathcal{L}(w^E, \lambda, \eta)$'s with $E \leqslant C$. Taking global sections, we see that the composition factors of $M = \Gamma(X, \mathcal{M}(w^C, \lambda, \eta))$ consist of some $\Gamma(X, \mathcal{L}(w^E, \lambda, \eta))$'s that are nonzero and with $E \leqslant C$. On the other hand, $L = \Gamma(X, \mathcal{L}(w^D, \lambda, \eta))$ is a composition factor of M. Hence $\Gamma(X, \mathcal{L}(w^D, \lambda, \eta)) = \Gamma(X, \mathcal{L}(w^E, \lambda, \eta))$ for some $E \leqslant C$. By the same uniqueness statement appeared in the preceding paragraph, $\mathcal{L}(w^D, \lambda, \eta) = \mathcal{L}(w^E, \lambda, \eta)$ and hence $D = E \leqslant C$. By the minimality of C, $D = C$. Thus $C = D$ is the minimum element in $W_\Theta \backslash W_\Theta v W^\lambda$ and $\Gamma(X, \mathcal{L}(w^C, \lambda, \eta)) = L$. □

We can pick a scalar $c \in \mathbb{C}$ so that $W^\lambda = W_{c\lambda}$. Then by 2.3.3, the set

$$A_\Theta^\lambda := A_{c\lambda} \cap (w_\Theta {}^\Theta W)$$

is a cross-section of $W_\Theta \backslash W / W^\lambda$ consisting of the unique shortest elements in each double coset. 5.2.3 can be rephrased as follows.

Corollary 5.2.4. *Let $\lambda \in \mathfrak{h}^*$ be antidominant and let $\eta : \mathfrak{n} \to \mathbb{C}$ be a character. Let $C \in W_\Theta \backslash W$. The following are equivalent:*

(a) $C = W_\Theta v$ *for some* $v \in A_\Theta^\lambda$;

(b) $\Gamma(X, \mathcal{L}(w^C, \lambda, \eta)) \neq 0$;

(c) $\Gamma(X, \mathcal{L}(w^C, \lambda, \eta)) = L(w^C\lambda, \eta)$.

Using these observations, we can write down a character formula for general infinitesimal characters.

Theorem 5.2.5 (Character formula: general case)**.** *Let $\lambda \in \mathfrak{h}^*$ be antidominant. Let $\eta : \mathfrak{n} \to \mathbb{C}$ be any character. For any $v \in A_\Theta^\lambda$, let $C = W_\Theta v$, and let $u \in A_{\Theta,\lambda}$ be the unique element such that $C \subseteq W_\Theta u W_\lambda$.*

Then

$$\operatorname{ch} L(\nu\lambda,\eta) = \operatorname{ch} L(w^C\lambda,\eta) = \sum_{z\in A_\Theta^\lambda \cap (W_\Theta u W_\lambda)} \left(\sum_{\substack{D\in W_\Theta\backslash W_\Theta z W^\lambda \\ D\leqslant_{u,\lambda} C}} P_{CD}^{u,\lambda}(-1) \right) \operatorname{ch} M(z\lambda,\eta), \qquad (5.2.6)$$

where the $P_{CD}^{u,\lambda}$*'s are parabolic Kazhdan-Lusztig polynomials for* $(W_\lambda, \Pi_\lambda, \Theta(u,\lambda))$ *as defined in 4.1.2. As* ν *ranges over* A_Θ^λ, $L(\nu\lambda,\eta)$ *exhausts all irreducible objects in* $\mathcal{N}_{\theta,\eta}$.

Proof. The right hand side is obtained by grouping the right side of (5.2.1) based on 5.2.2. In more detail, the cosets $W_\Theta \nu W^\lambda$ that are contained in $W_\Theta u W_\lambda$ partition $W_\Theta u W_\lambda$, and $A_\Theta^\lambda \cap (W_\Theta u W_\lambda)$ is a cross-section for this partition. We are simply grouping those standard modules within the same (W_Θ, W^λ)-cosets together. The left hand side and the last statement (that those $L(\nu\lambda,\eta)$'s exhaust all irreducibles) follows from 5.2.4 and Beilinson-Bernstein's equivalence of categories in the singular case. □

Appendix A

Examples

A.1 An A_2 example

Let us demonstrate the Kazhdan-Lusztig algorithm 4.2.2(1)(2)(3) in an A_2 example.

Figure A.1 through **Figure A.3** describe the A_2 root systems the A_2 Weyl group combinatorics for $\Theta = \{\alpha\}$ and three different choices of λ's. These three figures are related by non-integral intertwining functors:

$$\textbf{Figure A.1} \xleftrightarrow{I_{s_\beta}} \textbf{Figure A.2} \xleftrightarrow{I_{s_\alpha}} \textbf{Figure A.3}.$$

For example, in **Figure A.3**, the right W_Θ-cosets are pairs of elements connected by double lines: $W_\Theta = \{1, s_\alpha\}$, $W_\Theta s_\beta = \{s_\alpha s_\beta, s_\beta\}$, $W_\Theta s_\beta s_\alpha = \{s_\gamma, s_\beta s_\alpha\}$. The double (W_Θ, W_λ)-cosets are identified by looking at whether the Weyl group elements are underlined: elements in $W_\Theta W_\lambda = \{1, s_\alpha, s_\beta, s_\alpha s_\beta\}$ are underlined, and elements in $W_\Theta s_\beta s_\alpha W_\lambda = \{s_\beta s_\alpha, s_\gamma\}$ are not. For the single line connecting $s_\alpha s_\beta$ and s_α, this indicates the fact that $\mathcal{L}(s_\alpha s_\beta, -\frac{1}{2}\beta, \eta)$ is a direct summand of $U_\beta \mathcal{L}(s_\alpha, -\frac{1}{2}\beta, \eta)$, which is the geometric counterpart of the T_β part of the algorithm; after applying I_{s_α}, this relation between $\mathcal{L}(s_\alpha s_\beta, -\frac{1}{2}\beta, \eta)$ and $\mathcal{L}(s_\alpha, -\frac{1}{2}\beta, \eta)$ is translated to a relation between $\mathcal{L}(s_\gamma, -\frac{1}{2}\gamma, \eta)$ and $\mathcal{L}(s_\alpha, -\frac{1}{2}\gamma, \eta)$, which is indicated by the dotted line in **Figure A.2** connecting s_γ and s_α. Finally, if we look at all three diagrams, the elements s_γ in **Figure A.3**, $s_\alpha s_\beta$ in **Figure A.2**, and s_α in **Figure A.1** are all circled, which indicates the fact that $\mathcal{L}(s_\gamma, -\frac{1}{2}\beta, \eta) = I_{s_\alpha} \mathcal{L}(s_\alpha s_\beta, -\frac{1}{2}\gamma, \eta) = I_{s_\alpha} I_{s_\beta} \mathcal{L}(s_\alpha, -\frac{1}{2}\alpha, \eta)$; similarly for the boxed and hexed elements.

Note that the λ's in these diagrams are in the same Weyl group orbit θ, and only $\lambda = -\frac{1}{2}\gamma$ in **Figure A.2** is antidominant. In fact, if we only want to write down character formulas of irreducible modules in $\mathcal{N}_{\theta,\eta}$ based on Theorem 5.1.2, it is enough to look at **Figure A.2** alone. However, if one wants to run the Kazhdan-Lusztig algorithm 4.2.2, we need to look at all three diagrams.

Let us run the algorithm on these examples. First, we look at the smallest right W_Θ-cosets. The

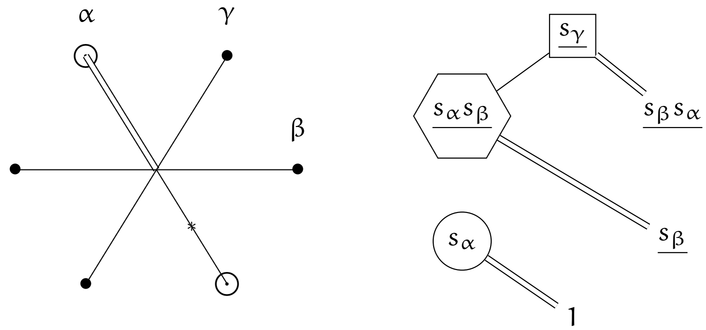


Figure A.1: The A_2 root system, $\Theta = \{\alpha\}$, $\lambda = -\frac{1}{2}\alpha$

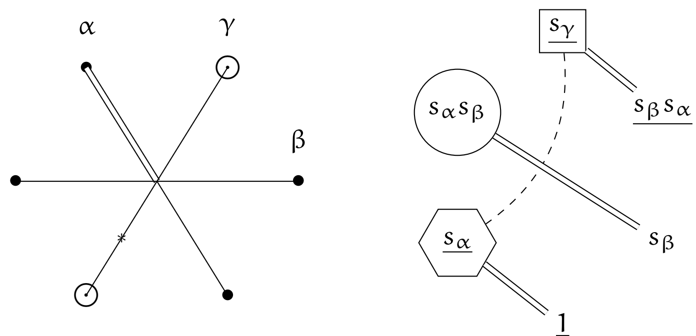


Figure A.2: The A_2 root system, $\Theta = \{\alpha\}$, $\lambda = -\frac{1}{2}\gamma$

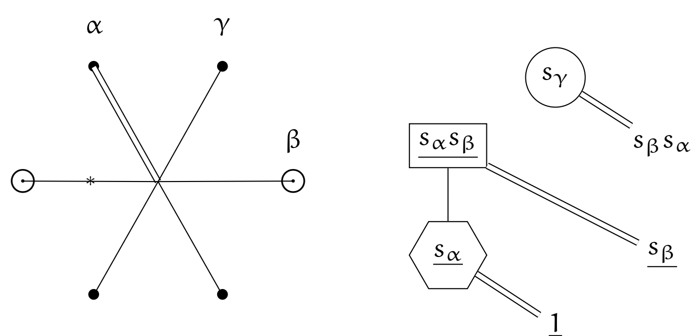


Figure A.3: The A_2 root system, $\Theta = \{\alpha\}$, $\lambda = -\frac{1}{2}\beta$

The diagrams on the left depict the root systems. The simple roots are α and β. $\Theta = \{\alpha\}$ which is indicated by a double line. λ is marked by $*$. Roots in Σ_λ are marked by $\odot$, and roots not in Σ_λ are marked by $\bullet$.

The diagrams on the right describe the Weyl group and combinatorics of double cosets. Nodes connected by double lines are in the same right W_Θ-coset. Nodes that are surrounded by a shape are the longest elements in right W_Θ-cosets. These are the elements that parametrize irreducible modules on X. Each diagram contains two double (W_Θ, W_λ)-cosets, one has four elements (underlined) and the other has two elements (not underlined). A single solid line indicates a pair of elements related by a U-functor. A single dotted line means that, after applying some non-integral intertwining functors, the pair of Weyl group elements are related by the U-functor for some other λ. Across all three diagrams, if two elements are surrounded by the same shape, then the irreducible/standard modules they correspond are sent to each other under some non-integral intertwining functors.

irreducible modules corresponding to the longest elements in these right cosets are equal to the standard modules containing them:

- in **Figure A.1**,

$$\mathcal{L}(s_\alpha, -\tfrac{1}{2}\alpha, \eta) = \mathcal{I}(s_\alpha, -\tfrac{1}{2}\alpha, \eta),$$
$$\varphi_{-\frac{1}{2}\alpha}(W_\Theta s_\alpha) = \delta_{W_\Theta s_\alpha};$$

- in **Figure A.2**,

$$\mathcal{L}(s_\alpha, -\tfrac{1}{2}\gamma, \eta) = \mathcal{I}(s_\alpha, -\tfrac{1}{2}\gamma, \eta),$$
$$\varphi_{-\frac{1}{2}\gamma}(W_\Theta s_\alpha) = \delta_{W_\Theta s_\alpha},$$
$$\operatorname{ch} L(-\tfrac{1}{2}s_\alpha\gamma, \eta) = \operatorname{ch} M(-\tfrac{1}{2}s_\alpha\gamma, \eta);$$

- in **Figure A.3**,

$$\mathcal{L}(s_\alpha, -\tfrac{1}{2}\beta, \eta) = \mathcal{I}(s_\alpha, -\tfrac{1}{2}\beta, \eta),$$
$$\varphi_{-\frac{1}{2}\beta}(W_\Theta s_\alpha) = \delta_{W_\Theta s_\alpha}.$$

Now we look at the second-to-smallest right W_Θ-cosets. Depending on the situation, we either apply U-functor or non-integral intertwining functor.

- Looking at $s_\alpha s_\beta$ in **Figure A.1**, s_β is the only simple reflection that descreases the length of the right coset $W_\Theta s_\alpha s_\beta$ s_β: $W_\Theta s_\alpha = W_\Theta s_\alpha s_\beta \cdot s_\beta < W_\Theta s_\alpha s_\beta$. Since s_β is non-integral to $\lambda = -\frac{1}{2}\alpha$, we apply I_{s_β} and we get

$$I_{s_\beta}\mathcal{L}(s_\alpha, -\tfrac{1}{2}\gamma, \eta) = \mathcal{L}(s_\alpha s_\beta, -\tfrac{1}{2}\alpha, \eta)$$

  by Corollary 3.2.11. This can be read off from the diagrams by noting that both $s_\alpha s_\beta$ in **Figure A.1** and s_α in **Figure A.2** are hexed. Since s_α in **Figure A.2** is in the lowest right W_Θ-coset, the corresponding standard module is irreducible. Hence the same is true for the standard module corresponding to $s_\alpha s_\beta$ in **Figure A.1**. More precisely, since we already know that $\mathcal{L}(s_\alpha, -\frac{1}{2}\gamma, \eta) = \mathcal{I}(s_\alpha, -\frac{1}{2}\gamma, \eta)$ and I_{s_β} is an equivalence of categories, the same relation is true for their images under I_{s_β}, i.e.

$$\mathcal{L}(s_\alpha s_\beta, -\tfrac{1}{2}\alpha, \eta) = \mathcal{I}(s_\alpha s_\beta, -\tfrac{1}{2}\alpha, \eta),$$
$$\varphi_{-\frac{1}{2}\alpha}(W_\Theta s_\alpha s_\beta) = \delta_{W_\Theta s_\alpha s_\beta}.$$

  Alternatively, one can find $\varphi_{-\frac{1}{2}\alpha}(W_\Theta s_\alpha s_\beta)$ purely combinatorially by using 4.2.2(3):

$$\varphi_{-\frac{1}{2}\alpha}(W_\Theta s_\alpha s_\beta) = \varphi_{-\frac{1}{2}\gamma}(W_\Theta s_\alpha) \cdots s_\beta = \delta_{W_\Theta s_\alpha} \cdot s_\beta = \delta_{W_\Theta s_\alpha s_\beta}.$$

- Looking at $s_\alpha s_\beta$ in **Figure A.2**, s_β is non-integral to $\lambda = -\frac{1}{2}\gamma$ and $W_\Theta s_\alpha = W_\Theta s_\alpha s_\beta \cdot s_\beta < W_\Theta s_\alpha s_\beta$. Hence we apply I_{s_β} which moves $s_\alpha s_\beta$ in **Figure A.2** to s_α in **Figure A.1**

$$\begin{aligned}\mathcal{L}(s_\alpha s_\beta, -\tfrac{1}{2}\gamma, \eta) &= I_{s_\beta}\mathcal{L}(s_\alpha, -\tfrac{1}{2}\alpha, \eta)\\ &= I_{s_\beta}\mathcal{I}(s_\alpha, -\tfrac{1}{2}\alpha, \eta)\\ &= \mathcal{I}(s_\alpha s_\beta, -\tfrac{1}{2}\gamma, \eta).\end{aligned}$$

  Hence

$$\begin{aligned}\varphi_{-\frac{1}{2}\gamma}(W_\Theta s_\alpha s_\beta) &= \delta_{W_\Theta s_\alpha s_\beta},\\ \operatorname{ch} L(-\tfrac{1}{2}s_\alpha s_\beta \gamma, \eta) &= \operatorname{ch} M(-\tfrac{1}{2}s_\alpha s_\beta\gamma, \eta).\end{aligned}$$

  This can be read off from the diagram: both $s_\alpha s_\beta$ in **Figure A.2** and s_α in **Figure A.1** are circled. Again, $\varphi_{-\frac{1}{2}\gamma}(W_\Theta s_\alpha s_\beta)$ can be found combinatorially by using 4.2.2(3):

$$\varphi_{-\frac{1}{2}\gamma}(W_\Theta s_\alpha s_\beta) = \varphi_{-\frac{1}{2}\alpha}(W_\Theta s_\alpha)\cdot s_\beta = \delta_{W_\Theta s_\alpha}\cdot s_\beta = \delta_{W_\Theta s_\alpha s_\beta}.$$

- Looking at $s_\alpha s_\beta$ in **Figure A.3**, s_β is integral to $\lambda = -\frac{1}{2}\beta$ and $W_\Theta s_\alpha = W_\Theta s_\alpha s_\beta \cdot s_\beta < W_\Theta s_\alpha s_\beta$. Hence

$$\mathcal{L}(s_\alpha s_\beta, -\tfrac{1}{2}\beta, \eta) \text{ is a direct summand of } U_\beta \mathcal{L}(s_\alpha, -\tfrac{1}{2}\beta, \eta).$$

  This is also indicated by the single solid line connecting s_α and $s_\alpha s_\beta$. By Theorem 4.2.2(2), there are integers c_{s_α} and $c_{s_\alpha s_\beta}$ such that

$$T_\beta(\varphi_{-\frac{1}{2}\beta}(W_\Theta s_\alpha)) = c_{s_\alpha}\varphi_{-\frac{1}{2}\beta}(W_\Theta s_\alpha) + c_{s_\alpha s_\beta}\varphi_{-\frac{1}{2}\beta}(W_\Theta s_\alpha s_\beta).$$

  Using the fact that $\varphi_{-\frac{1}{2}\beta}(W_\Theta s_\alpha) = \delta_{W_\Theta s_\alpha}$ and using the definition of T_β, this equation becomes

$$q\delta_{W_\Theta s_\alpha} + \delta_{W_\Theta s_\alpha s_\beta} = T_\beta(\delta_{W_\Theta s_\alpha}) = c_{s_\alpha}\delta_{W_\Theta s_\alpha} + c_{s_\alpha s_\beta}\varphi_{-\frac{1}{2}\beta}(W_\Theta s_\alpha s_\beta).$$

  Also, from Theorem 4.2.2(1), $\varphi_{-\frac{1}{2}\beta}(W_\Theta s_\alpha s_\beta) = \delta_{W_\Theta s_\alpha s_\beta} + P\delta_{W_\Theta s_\alpha}$ for some $P \in q\mathbb{Z}[q]$. The above equation then becomes

$$q\delta_{W_\Theta s_\alpha} + \delta_{W_\Theta s_\alpha s_\beta} = (c_{s_\alpha} + P)\delta_{W_\Theta s_\alpha} + c_{s_\alpha s_\beta}\delta_{W_\Theta s_\alpha}.$$

  Comparing coefficients on both sides, we see that $c_{s_\alpha} = 0$, $P = q$, and $c_{s_\alpha s_\beta} = 1$. Hence

$$\begin{aligned}\varphi_{-\frac{1}{2}\beta}(W_\Theta s_\alpha s_\beta) &= \delta_{W_\Theta s_\alpha s_\beta} + q\delta_{W_\Theta s_\alpha},\\ [\mathcal{L}(s_\alpha s_\beta, -\tfrac{1}{2}\beta, \eta)] &= [\mathcal{I}(s_\alpha s_\beta, -\tfrac{1}{2}\beta, \eta)] - [\mathcal{I}(s_\alpha, -\tfrac{1}{2}\beta, \eta)].\end{aligned}$$

Now we look at the largest right W_Θ-cosets.

- Looking at s_γ in **Figure A.1**, $\mathcal{L}(s_\gamma, -\frac{1}{2}\alpha, \eta)$ is a direct summand of $U_\alpha \mathcal{L}(s_\alpha s_\beta, -\frac{1}{2}\alpha, \eta)$. Hence,

$$\begin{aligned} q\delta_{W_\Theta s_\alpha s_\beta} + \delta_{W_\Theta s_\gamma} &= T_\alpha(\delta_{W_\Theta s_\alpha s_\beta}) \\ &= T_\alpha(\varphi_{-\frac{1}{2}\alpha}(W_\Theta s_\alpha s_\beta)) \\ &= c_{s_\alpha}\varphi_{-\frac{1}{2}\alpha}(W_\Theta s_\alpha) + c_{s_\alpha s_\beta}\varphi_{-\frac{1}{2}\alpha}(W_\Theta s_\alpha s_\beta) + c_{s_\gamma}\varphi_{-\frac{1}{2}\alpha}(W_\Theta s_\gamma) \\ &= c_{s_\alpha}\delta_{W_\Theta s_\alpha} + c_{s_\alpha s_\beta}\delta_{W_\Theta s_\alpha s_\beta} + c_{s_\gamma}(\delta_{W_\Theta s_\gamma} + P\delta_{W_\Theta s_\alpha s_\beta}) \end{aligned}$$

  for some integers c's and some polynomial $P \in q\mathbb{Z}[q]$. Here the first equality is by definition of T_α; the second equality is because $\varphi_{-\frac{1}{2}\alpha}(W_\Theta s_\alpha s_\beta) = \delta_{W_\Theta s_\alpha s_\beta}$ which was computed before; the third equality is by 4.2.2(2); the last equality follows from replacing $\varphi_{-\frac{1}{2}\alpha}(W_\Theta s_\alpha)$ and $\varphi_{-\frac{1}{2}\alpha}(W_\Theta s_\alpha s_\beta)$ by their expressions in the δ's (which were already computed before) and rewriting $\varphi_{-\frac{1}{2}\alpha}(W_\Theta s_\gamma)$ by 4.2.2(1). Comparing the coefficients on both sides, we see that $c_{s_\alpha} = c_{s_\alpha s_\beta} = 0$, $c_{s_\gamma} = 1$, and $P = q$. Hence

$$\begin{aligned} &\varphi_{-\frac{1}{2}\alpha}(W_\Theta s_\gamma) = \delta_{W_\Theta s_\gamma} + q\delta_{W_\Theta s_\alpha s_\beta}, \\ &[\mathcal{L}(s_\gamma, -\tfrac{1}{2}\alpha, \eta)] = [\mathcal{I}(s_\gamma, -\tfrac{1}{2}\alpha, \eta)] - [\mathcal{I}(s_\alpha s_\beta, -\tfrac{1}{2}\alpha, \eta)]. \end{aligned}$$

- For s_γ in **Figure A.2**, by applying I_{s_α},

$$\mathcal{L}(s_\gamma, -\tfrac{1}{2}\gamma, \eta) = I_{s_\alpha}\mathcal{L}(s_\alpha s_\beta, -\tfrac{1}{2}\beta, \eta).$$

  Hence, using 4.2.2(3),

$$\begin{aligned} &\varphi_{-\frac{1}{2}\gamma}(W_\Theta s_\gamma) = \varphi_{-\frac{1}{2}\beta}(W_\Theta s_\alpha s_\beta)s_\alpha = \delta_{W_\Theta s_\gamma} + q\delta_{W_\Theta s_\alpha}, \\ &[\mathcal{L}(s_\gamma, -\tfrac{1}{2}\gamma, \eta)] = [\mathcal{I}(s_\gamma, -\tfrac{1}{2}\gamma, \eta)] - [\mathcal{I}(s_\alpha, -\tfrac{1}{2}\gamma, \eta)], \\ &\operatorname{ch} L(-\tfrac{1}{2}s_\gamma\gamma, \eta) = \operatorname{ch} M(-\tfrac{1}{2}s_\gamma\gamma, \eta) - \operatorname{ch} M(-\tfrac{1}{2}s_\alpha\gamma, \eta). \end{aligned}$$

- For s_γ in **Figure A.3**, again by using 4.2.2(3),

$$\begin{aligned} &\varphi_{-\frac{1}{2}\beta}(W_\Theta s_\gamma) = \varphi_{-\frac{1}{2}\gamma}(W_\Theta s_\alpha s_\beta)s_\alpha = \delta_{W_\Theta s_\gamma}, \\ &[\mathcal{L}(s_\gamma, -\tfrac{1}{2}\beta, \eta)] = [\mathcal{I}(s_\gamma, -\tfrac{1}{2}\beta, \eta)]. \end{aligned}$$

A.2 An A_3 example

The A_3 root system is the smallest example in which all nontrivial phenomena appear. Let us apply the character formula 5.1.2 to this example.

The root system, integral roots, and roots in Θ are shown in **Figure A.4** on the facing page. Here the integral subsystem is of type A_2. **Figure A.5** on the next page is a diagram of the A_3 Weyl group, arranged in a way so that elements in the same right W_Θ-coset are grouped together and are connected by double lines. There are two double (W_Θ, W_λ)-cosets: $W_\Theta s_\gamma s_\beta W_\lambda$ and $W_\Theta W_\lambda$. Elements in $W_\Theta s_\gamma s_\beta W_\lambda$ are underlined.

Let's first look at the double coset $W_\Theta s_\gamma s_\beta W_\lambda$, with $u = s_\gamma s_\beta \in A_{\Theta,\lambda}$. It equals a single left W_λ-coset $s_\gamma s_\beta W_\lambda$ and a single right W_Θ-coset $W_\Theta s_\gamma s_\beta$. Hence

$$\Theta(s_\gamma s_\beta, \lambda) = \Pi_\lambda, \quad W_{\lambda,\Theta(s_\gamma s_\beta,\lambda)} = W_\lambda,$$

and $(W_\Theta s_\gamma s_\beta)|_\lambda = W_\lambda 1$, the unique right W_λ-coset in W_λ. Therefore

$$\varphi_\lambda(W_\Theta s_\gamma s_\beta) = \delta_{W_\Theta s_\gamma s_\beta},$$
$$\operatorname{ch} L(s_\gamma s_\beta \lambda, \eta) = \operatorname{ch} M(s_\gamma s_\beta \lambda, \eta).$$

Now let's look at the other double coset $W_\Theta W_\lambda$, with $u = 1$ and

$$\Theta(1, \lambda) = \{\alpha + \beta\}, \quad W_{\lambda,\Theta(1,\lambda)} = \{1, s_{\alpha+\beta}\}.$$

For convenience, we write

$$W_\bullet := W_{\lambda,\Theta(1,\lambda)}.$$

The root system Σ_λ and the Weyl group for $(W_\lambda, \Pi_\lambda, \Theta(1, \lambda))$ is shown in **Figure A.6** on page 82. The map $(-)|_\lambda$ restricted to $W_\Theta \backslash W_\Theta W_\lambda$ can be visualized as in **Figure A.7** on page 82, where a coset on the right hand side is sent to the coset on the left with the same shape.

The Whittaker Kazhdan-Lusztig polynomials for $(W_\lambda, \Pi_\lambda, \Theta(1, \lambda))$ are shown in **Table A.1** on this page.

$P^{1,\lambda}_{EF}$	$W_\bullet s_{\alpha+\beta}$	$W_\bullet s_{\alpha+\beta} s_\gamma$	$W_\bullet s_{\alpha+\beta+\gamma}$
$W_\bullet s_{\alpha+\beta}$	1	0	0
$W_\bullet s_{\alpha+\beta} s_\gamma$	q	1	0
$W_\bullet s_{\alpha+\beta+\gamma}$	0	q	1

Table A.1: Whittaker Kazhdan-Lusztig polynomials for $(W_\lambda, \Pi_\lambda, \Theta(1, \lambda))$

Hence, our Theorem 4.2.2(1)(4) says

$$\begin{aligned}\varphi_\lambda(W_\Theta s_\alpha s_\beta s_\alpha) = {} & P^{1,\lambda}_{(W_\bullet s_{\alpha+\beta}),(W_\bullet s_{\alpha+\beta})} \delta_{W_\Theta s_\alpha s_\beta s_\alpha} \\ & + P^{1,\lambda}_{(W_\bullet s_{\alpha+\beta}),(W_\bullet s_{\alpha+\beta} s_\gamma)} \delta_{W_\Theta s_\alpha s_\beta s_\alpha s_\gamma} \\ & + P^{1,\lambda}_{(W_\bullet s_{\alpha+\beta}),(W_\bullet s_{\alpha+\beta+\gamma})} \delta_{W_\Theta w_0}\end{aligned}$$

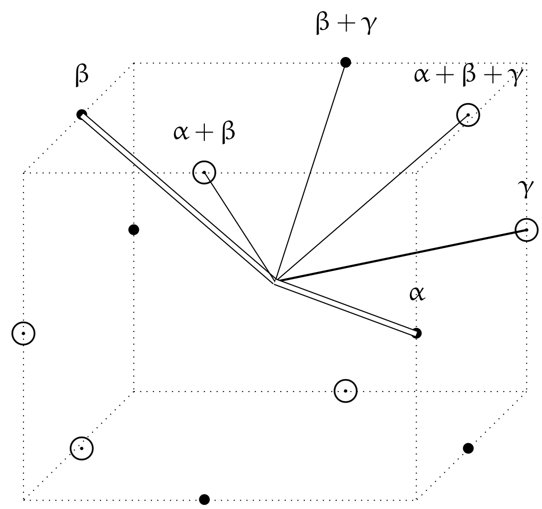


Figure A.4: The A_3 root system.
The set of simple roots is $\Pi = \{\alpha, \beta, \gamma\}$. $\Theta = \{\alpha, \beta\}$, indicated by double lines. Roots in Σ_λ are marked by $\odot$, and those not in Σ_λ are marked by $\bullet$. To make the picture more readable, only the positive roots are connected to the origin.
Here λ is chosen to be $\lambda = -m\rho + c(-\alpha + 2\beta + \gamma)$ for any nonzero number c transcendental over $\mathbb{Q}$ and any large enough integer m so that λ is antidominant regular (note that $-\alpha + 2\beta + \gamma$ is a vector perpendicular to the plane spanned by $\alpha + \beta$ and γ).

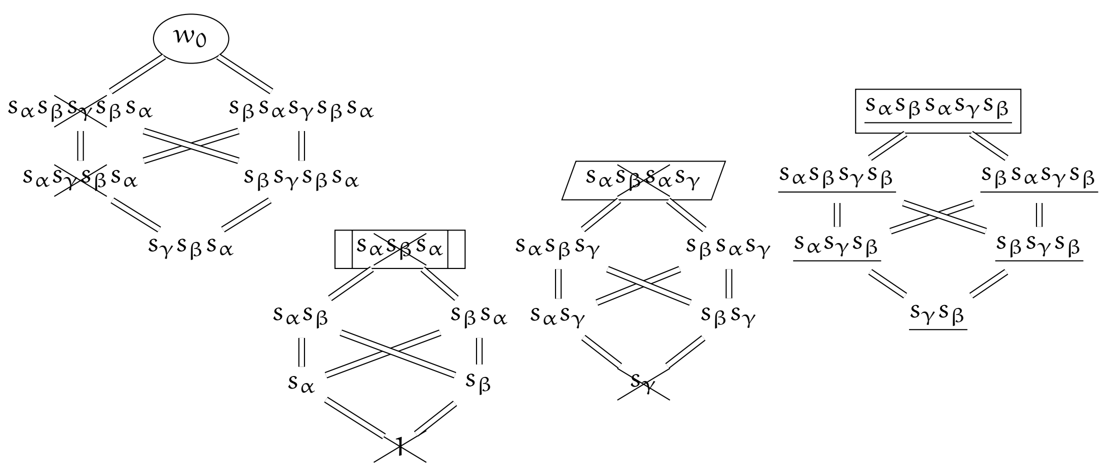


Figure A.5: Double cosets in the A_3 Weyl group.
Elements surrounded by various shapes are the longest elements in right W_Θ-cosets. Elements that are crossed out are those in W_λ. Elements in $W_\Theta s_\gamma s_\beta W_\lambda$ are underlined.

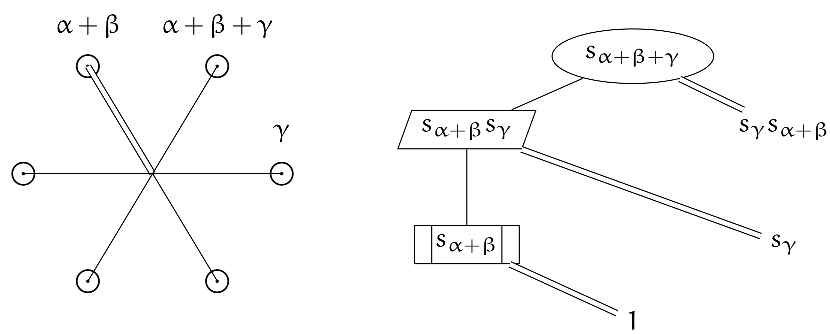


Figure A.6: The A_2 integral roots and its Weyl group

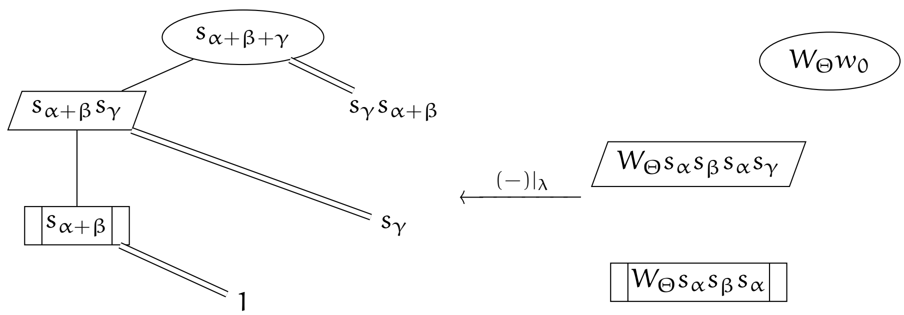


Figure A.7: $(-)|_\lambda$ on right cosets

$$
\begin{aligned}
&= \delta_{W_\Theta s_\alpha s_\beta s_\alpha}, \\
\varphi_\lambda(W_\Theta s_\alpha s_\beta s_\alpha s_\gamma) &= P^{1,\lambda}_{(W_\bullet s_{\alpha+\beta} s_\gamma),(W_\bullet s_{\alpha+\beta})} \delta_{W_\Theta s_\alpha s_\beta s_\alpha} \\
&\quad + P^{1,\lambda}_{(W_\bullet s_{\alpha+\beta} s_\gamma),(W_\bullet s_{\alpha+\beta} s_\gamma)} \delta_{W_\Theta s_\alpha s_\beta s_\alpha s_\gamma} \\
&\quad + P^{1,\lambda}_{(W_\bullet s_{\alpha+\beta} s_\gamma),(W_\bullet s_{\alpha+\beta+\gamma})} \delta_{W_\Theta w_0} \\
&= q\delta_{W_\Theta s_\alpha s_\beta s_\alpha} + \delta_{W_\Theta s_\alpha s_\beta s_\alpha s_\gamma}, \\
\varphi_\lambda(W_\Theta w_0) &= P^{1,\lambda}_{(W_\bullet s_{\alpha+\beta+\gamma}),(W_\bullet s_{\alpha+\beta})} \delta_{W_\Theta s_\alpha s_\beta s_\alpha} \\
&\quad + P^{1,\lambda}_{(W_\bullet s_{\alpha+\beta+\gamma}),(W_\bullet s_{\alpha+\beta} s_\gamma)} \delta_{W_\Theta s_\alpha s_\beta s_\alpha s_\gamma} \\
&\quad + P^{1,\lambda}_{(W_\bullet s_{\alpha+\beta+\gamma}),(W_\bullet s_{\alpha+\beta+\gamma})} \delta_{W_\Theta w_0} \\
&= q\delta_{W_\Theta s_\alpha s_\beta s_\alpha s_\gamma} + \delta_{W_\Theta w_0}.
\end{aligned}
$$

Specializing to $q = -1$, we get

$$
\begin{aligned}
\operatorname{ch} L(s_\alpha s_\beta s_\alpha \lambda, \eta) &= \operatorname{ch} M(s_\alpha s_\beta s_\alpha \lambda, \eta), \\
\operatorname{ch} L(s_\alpha s_\beta s_\alpha s_\gamma \lambda, \eta) &= -\operatorname{ch} M(s_\alpha s_\beta s_\alpha \lambda, \eta) + \operatorname{ch} M(s_\alpha s_\beta s_\alpha s_\gamma \lambda, \eta), \\
\operatorname{ch} L(w_0 \lambda, \eta) &= -\operatorname{ch} M(s_\alpha s_\beta s_\alpha s_\gamma \lambda, \eta) + \operatorname{ch} M(w_0 \lambda, \eta).
\end{aligned}
$$

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List of Notations

Index