Ubiquity of Littlewood-Richardson Coefficients

Cameron Zhao

Department of Mathematics, University of Utah

October 4, 2019

mathoverflow

Home
Questions
Tags
Users
Unanswered

Your favorite surprising connections in Mathematics

173

177

There are certain things in mathematics that have caused me a pleasant surprise -- when some part of mathematics is brought to bear in a fundamental way on another, where the connection between the two is unexpected. The first example that comes to my mind is the proof by Furstenberg and Katznelson of Szemeredi's theorem on the existence of arbitrarily long arithmetic progressions in a set of integers which has positive upper Banach density, but using ergodic theory. Of course in the years since then, this idea has now become enshrined and may no longer be viewed as surprising, but it certainly was when it was first devised.

Another unexpected connection was when Kolmogorov used Shannon's notion of probabilistic entropy as an important invariant in dynamical systems.

So, what other surprising connections are there out there?

soft-question big-list big-picture

share cite improve this question

edited Aug 23 '15 at 17:26

community wiki
5 revs, 3 users 100%
Victor Miller

44

The ubiquity of Littlewood-Richardson coefficients. Given three partitions λ, μ, ν each with at most n parts, there is a combinatorial definition for a number $c^{\nu}_{\lambda,\mu}$ which is nonzero if and only if any of the following statements are true:

- There exist Hermitian matrices A, B, C whose eigenvalues are λ, μ, ν, respectively and $A + B = C$ (one can also replace Hermitian by real symmetric)
- The irreducible representation of $\mathbf{GL}_n(\mathbf{C})$ with highest weight ν is a subrepresentation of the tensor product of those irreducible representations with highest weights λ and μ.
- Indexing the Schubert cells of the Grassmannian $\mathbf{Gr}(d, \mathbf{C}^m)$ (where $d \geq n$ and $m - d$ is at least as big as any part of λ, μ, ν) by σ_λ appropriately, the cycle σ_ν appears in the intersection product $\sigma_\lambda \sigma_\mu$.
- There exists finite Abelian p-groups A, B, C and a short exact sequence $0 \to A \to B \to C \to 0$ such that $B \cong \bigoplus_i \mathbf{Z}/p^{\nu_i}$, $A \cong \bigoplus_i \mathbf{Z}/p^{\lambda_i}$, and $C \cong \bigoplus_i \mathbf{Z}/p^{\mu_i}$.

And probably many more things.

share cite improve this answer

answered Feb 8 '10 at 5:54

community wiki
Steven Sam

Table of Contents

1. Defining LR Coefficients
2. Decomposing Representations of $\mathbf{GL}_n$
3. Eigenvalues of Hermitian Matrices
4. Extensions of Finite Abelian p-Groups
5. Intersection of Schubert Cells
6. Underlying Connections

Table of Contents

1 Defining LR Coefficients

2 Decomposing Representations of $\mathbf{GL}_n$

3 Eigenvalues of Hermitian Matrices

4 Extensions of Finite Abelian p-Groups

5 Intersection of Schubert Cells

6 Underlying Connections

Combinatorial Definition of LR Coefficients

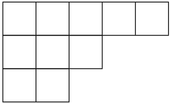
Definition (Paritions and Young diagrams)

A partition of $n \in \mathbb{N}_+$ is a weakly decreasing integer sequence $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots)$ that sums up to n, i.e. $\sum_i \lambda_i = n$. Denoted $\lambda \vdash n$. Define $|\lambda| = \sum_i \lambda_i$.
A Young diagram is a collection of finitely many "top-left aligned" boxes.

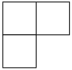
$$\{\text{Paritions of } n\} \overset{\sim}{\longleftrightarrow} \{\text{Young diagrams with } n \text{ boxes}\}.$$

Combinatorial Definition of LR Coefficients

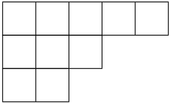
$$\lambda = (5, 3, 2) \vdash 10 \longleftrightarrow$$



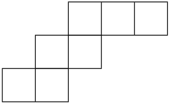
$$\mu =$$



$$\subset$$



$$= \lambda \rightsquigarrow \lambda \backslash \mu =$$



.

Combinatorial Definition of LR Coefficients

We can also fill in numbers

1	6	9	4	3
9	8	7		
6	1			

Combinatorial Definition of LR Coefficients

Let α, β, γ be Young diagrams (or partitions).

Combinatorial Definition of LR Coefficients

Let α, β, γ be Young diagrams (or partitions).

Definition (Littlewood-Richardson Coefficients)

The **LR coefficients** $c^{\gamma}_{\alpha,\beta}$ is defined as follows.

- If $\alpha \subseteq \gamma$ and $|\gamma \backslash \alpha| = |\beta|$, then $c^{\gamma}_{\alpha,\beta}$ is the number of ways to fill numbers into $\gamma \backslash \alpha$ such that:
    1. each row is weakly increasing, each column is strongly increasing;
    2. the number i appears β_i times; and
    3. if we concatenate the rows of $\gamma \backslash \alpha$ (start from the bottom row, end at top row), read the numbers from right to left, then a larger number should not appear less often or as often than a smaller number (i.e. a "lattice word").

(A filled diagram following these rules is called a LR tableau.)

Combinatorial Definition of LR Coefficients

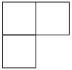
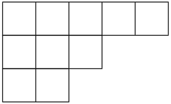
Definition (Littlewood-Richardson Coefficients)

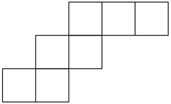
- Otherwise set $c^{\gamma}_{\alpha,\beta} = 0$.

Example

$$\alpha = \ydiagram{2,1}\,, \ \beta = (3,3,1), \ \gamma = \ydiagram{5,3,2}\,.$$

Example

$\alpha =$ , $\beta = (3, 3, 1)$, $\gamma =$ .

Fill $\{1, 1, 1, 2, 2, 2, 3\}$ in $\gamma \backslash \alpha =$ .

Example

		1	1	1
	2	2		
2	3			

is an LR tableau:

Example

		1	1	1
	2	2		
2	3			

is an LR tableau:

✓ each row is weakly increasing, each column is strongly increasing;

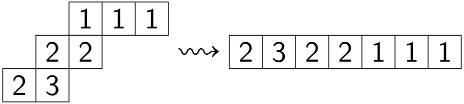
Example

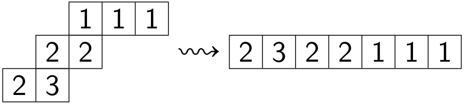
$$\begin{array}{|c|c|c|c|c|} \cline{3-5} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 1 & 1 & 1 \\ \cline{2-5} \multicolumn{1}{c|}{} & 2 & 2 & \multicolumn{2}{c}{} \\ \cline{1-3} 2 & 3 & \multicolumn{3}{c}{} \\ \cline{1-2} \end{array}$$

is an LR tableau:

- ✓ each row is weakly increasing, each column is strongly increasing;
- ✓ the number i appears β_i times ($\beta = (3, 3, 1)$);

		1	1	1
	2	2		
2	3			


1 1 1
2 2
2 3
2 3 2 2 1 1 1



Read from right to left:

1

11

111

2111

22111

322111

2322111

Example

$\alpha = (2, 1), \beta = (3, 3, 1), \gamma = (5, 3, 2)$.

It's easy to see that

		1	1	1
	2	2		
2	3			

is the only LR tableau of our triple (α, β, γ). Therefore $c^{\gamma}_{\alpha,\beta} = 1$.

Table of Contents

1 Defining LR Coefficients

2 Decomposing Representations of $\mathbf{GL}_n$

3 Eigenvalues of Hermitian Matrices

4 Extensions of Finite Abelian p-Groups

5 Intersection of Schubert Cells

6 Underlying Connections

History of LR coefficients

Historically the LR coefficients are not defined combinatorially, but actually in representation theoretic context. Littlewood and Richardson proved the combinatorial characterization of these coefficients.

Representations

A **representation** of a group G is a (finite dimensional) vector space V on which G acts by linear automorphisms.

Representations

A **representation** of a group G is a (finite dimensional) vector space V on which G acts by linear automorphisms.
For example, any vector space V is a representation of $\mathbf{GL}(V)$.

Examples of Representations

We've seen V is a representation of $\mathbf{GL}(V)$. It's irreducible.

Examples of Representations

We've seen V is a representation of $\mathbf{GL}(V)$. It's irreducible.
$V^{\otimes d}$ is also a representation of $\mathbf{GL}(V)$:

$$A \cdot (v_1 \otimes \cdots \otimes v_m) = (Av_1) \otimes \cdots \otimes (Av_m).$$

Not irreducible

Examples of Representations

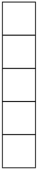
We've seen V is a representation of $\mathbf{GL}(V)$. It's irreducible.
$V^{\otimes d}$ is also a representation of $\mathbf{GL}(V)$:

$$A \cdot (v_1 \otimes \cdots \otimes v_m) = (Av_1) \otimes \cdots \otimes (Av_m).$$

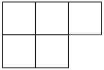
Not irreducible
$\mathrm{Sym}^d V = \mathbb{C}[e_1, \ldots, e_{\dim V}]_d$, $\bigwedge^d V$, both irreducible

Irreducible Representations of $\mathbf{GL}(V)$

$\mathrm{Sym}^5 V \leftrightsquigarrow$ 

$\bigwedge^5 V \leftrightsquigarrow$ .

Irreducible Representations of $\mathbf{GL}(V)$



$\leftrightsquigarrow$ "$(\mathrm{Sym}^3 V) \wedge (\mathrm{Sym}^2 V)$"?

"$(\bigwedge^2 V)\,\mathrm{Sym}(\bigwedge^2 V)\,\mathrm{Sym}(\bigwedge^1 V)$"?

Irreducible Representations of $\mathbf{GL}(V)$

There's a correct way to jazz up Sym and $\bigwedge$.

$$\leftrightsquigarrow S_{(3,2)}V.$$

$$\lambda \leftrightsquigarrow S_\lambda V.$$

Called **Schur modules**.

Irreducible Representations of $\mathbf{GL}(V)$

Theorem

Irreducible polynomial representations of $\mathbf{GL}(V)$ *are exactly the* $S_\lambda V$*'s, as* λ *ranges over partitions of length* $\leqslant \dim V$ *(i.e.* $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_{\dim V} \geqslant 0 \geqslant \cdots)$*).*

LR Coefficients in Representations of $\mathbf{GL}(V)$

We can tensor two irreps. This will in general result in a reducible representation.

LR Coefficients in Representations of $\mathbf{GL}(V)$

We can tensor two irreps. This will in general result in a reducible representation.

$$\rightsquigarrow S_\alpha V \otimes_{\mathbb{C}} S_\beta V = \bigoplus_\gamma (S_\gamma V)^{\oplus \boldsymbol{d}^\gamma_{\alpha,\beta}}$$

LR Coefficients in Representations of $\mathbf{GL}(V)$

We can tensor two irreps. This will in general result in a reducible representation.

$$\rightsquigarrow S_\alpha V \otimes_{\mathbb{C}} S_\beta V = \bigoplus_\gamma (S_\gamma V)^{\oplus \boldsymbol{d}^\gamma_{\alpha,\beta}}$$

Theorem

The multiplicities $d^\gamma_{\alpha,\beta}$ equals the LR coefficients $c^\gamma_{\alpha,\beta}$. In other words, $c^\gamma_{\alpha,\beta}$ is exactly the number of times $S_\gamma V$ appears in $S_\alpha V \otimes S_\beta V$.

LR Coefficients in Representations of $\mathbf{GL}(V)$

Using this, we can go back and forth between results on $c^{\gamma}_{\alpha,\beta}$ and results about the $S_\lambda V$'s.

LR Coefficients in Representations of $\mathbf{GL}(V)$

Using this, we can go back and forth between results on $c^{\gamma}_{\alpha,\beta}$ and results about the $S_\lambda V$'s.
For example, because $S_\alpha V \otimes S_\beta V \cong S_\beta V \otimes S_\alpha V$, we know $c^{\gamma}_{\alpha,\beta} = c^{\gamma}_{\beta,\alpha}$. This is not obvious from the definition of $c^{\gamma}_{\alpha,\beta}$.

Table of Contents

1 Defining LR Coefficients

2 Decomposing Representations of $\mathbf{GL}_n$

3 Eigenvalues of Hermitian Matrices

4 Extensions of Finite Abelian p-Groups

5 Intersection of Schubert Cells

6 Underlying Connections

Eigenvalues of Hermitian Matrices

We know that given an $n \times n$ Hermitian matrix A, it's eigenvalues are necessarily real. Let $\alpha = (\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_n)$ be its eigenvalues.

Eigenvalues of Hermitian Matrices

We know that given an $n \times n$ Hermitian matrix A, it's eigenvalues are necessarily real. Let $\alpha = (\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_n)$ be its eigenvalues.

Question

Given descending sequences α, β, γ of length n, are there Hermitian matrices A, B, C with eigenvalues α, β, γ so that $A + B = C$?

Necessary Conditions

- Obviously: $\sum_i \gamma_i = \sum_i \alpha_i + \sum_i \beta_i$.

Necessary Conditions

- Obviously: $\sum_i \gamma_i = \sum_i \alpha_i + \sum_i \beta_i$.
- More than a century ago: $\gamma_1 \leqslant \alpha_1 + \beta_1$.

Necessary Conditions

- Obviously: $\sum_i \gamma_i = \sum_i \alpha_i + \sum_i \beta_i$.
- More than a century ago: $\gamma_1 \leqslant \alpha_1 + \beta_1$.
- H. Weyl, 1912: $\forall\, i + j - 1 \leqslant n$, $\gamma_{i+j-1} \leqslant \alpha_i + \beta_j$.

Necessary Conditions

- Obviously: $\sum_i \gamma_i = \sum_i \alpha_i + \sum_i \beta_i$.
- More than a century ago: $\gamma_1 \leqslant \alpha_1 + \beta_1$.
- H. Weyl, 1912: $\forall\, i + j - 1 \leqslant n$, $\gamma_{i+j-1} \leqslant \alpha_i + \beta_j$.
- (These three conditions above are also sufficient when the matrices are 2×2.)

Necessary Conditions

- Obviously: $\sum_i \gamma_i = \sum_i \alpha_i + \sum_i \beta_i$.
- More than a century ago: $\gamma_1 \leqslant \alpha_1 + \beta_1$.
- H. Weyl, 1912: $\forall\, i + j - 1 \leqslant n$, $\gamma_{i+j-1} \leqslant \alpha_i + \beta_j$.
- (These three conditions above are also sufficient when the matrices are 2×2.)
- K. Fan, 1949: $\forall\, i < n$, $\sum_{i \leqslant r} \gamma_i \leqslant \sum_{i \leqslant r} \alpha_i + \sum_{i \leqslant r} \beta_i$.

Necessary Conditions

- Obviously: $\sum_i \gamma_i = \sum_i \alpha_i + \sum_i \beta_i$.
- More than a century ago: $\gamma_1 \leqslant \alpha_1 + \beta_1$.
- H. Weyl, 1912: $\forall\, i + j - 1 \leqslant n$, $\gamma_{i+j-1} \leqslant \alpha_i + \beta_j$.
- (These three conditions above are also sufficient when the matrices are 2×2.)
- K. Fan, 1949: $\forall\, i < n$, $\sum_{i \leqslant r} \gamma_i \leqslant \sum_{i \leqslant r} \alpha_i + \sum_{i \leqslant r} \beta_i$.
- V. B. Lidskii, 1950, and H. Wielandt, 1955: $\forall\, I \subsetneq \{1, \ldots, n\}$, $\sum_{i \in I} \gamma_i \leqslant \sum_{i \in I} \alpha_i + \sum_{i \leqslant |I|} \beta_i$.

Necessary Conditions

- Obviously: $\sum_i \gamma_i = \sum_i \alpha_i + \sum_i \beta_i$.
- More than a century ago: $\gamma_1 \leqslant \alpha_1 + \beta_1$.
- H. Weyl, 1912: $\forall\, i + j - 1 \leqslant n$, $\gamma_{i+j-1} \leqslant \alpha_i + \beta_j$.
- (These three conditions above are also sufficient when the matrices are 2×2.)
- K. Fan, 1949: $\forall\, i < n$, $\sum_{i \leqslant r} \gamma_i \leqslant \sum_{i \leqslant r} \alpha_i + \sum_{i \leqslant r} \beta_i$.
- V. B. Lidskii, 1950, and H. Wielandt, 1955: $\forall\, I \subsetneq \{1, \ldots, n\}$, $\sum_{i \in I} \gamma_i \leqslant \sum_{i \in I} \alpha_i + \sum_{i \leqslant |I|} \beta_i$.
- A. Horn, 1962: All conditions above, plus $\gamma_2 + \gamma_3 \leqslant \alpha_1 + \alpha_3 + \beta_1 + \beta_3$, are also sufficient for 3×3 matrices.

Necessary Conditions

Observe that all necessary conditions above have the form

$$\sum_{k \in K} \gamma_k \leqslant \sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j \qquad (*_{IJK})$$

for certain subsets I, J, K of $\{1, \ldots, n\}$.

Necessary Conditions

Observe that all necessary conditions above have the form

$$\sum_{k \in K} \gamma_k \leqslant \sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j \qquad (*_{IJK})$$

for certain subsets I, J, K of $\{1, \ldots, n\}$.

A. Horn, 1962: If we "collect" all conditions in the above form, do we get a necessary and sufficient condition?

Of course we have to carefully collect $(*_{IJK})$'s. Horn conjectured that the collection $\bigcup_{r<n} T_r$ works, where T_r is defined inductively:

Of course we have to carefully collect $(*_{IJK})$'s. Horn conjectured that the collection $\bigcup_{r<n} T_r$ works, where T_r is defined inductively:

$$T_1 := \{(I,J,K) \mid \sum I + \sum J = \sum K + 1\}$$

$$T_r := \left\{ (I,J,K) \mid \begin{array}{c} \sum I + \sum J = \sum K + \frac{r(r+1)}{2}; \\ \forall p < r, \forall (F,G,H) \in T_p, \\ \sum_{f\in F} i_f + \sum_{g\in G} j_g \leqslant \sum_{h\in H} k_h + \frac{p(p+1)}{2} \end{array} \right\}$$

Horn's Conjecture

Horn's Conjecture (proven by A. Knutson and T. Tao)

A triple (α, β, γ) occurs as eigenvalues of Hermitian matrices A, B and $C = A + B$ iff. the following two conditions hold

- $\sum_i \gamma_i = \sum_i \alpha_i + \sum_i \beta_i$, and
- the inequalities

$$\sum_{k \in K} \gamma_k \leqslant \sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j \qquad (*_{IJK})$$

holds for all (I, J, K) in T_r for all $r < n$.

Connections with LR Coefficients

At the level of combinatorics, the connection is:

Theorem (A. Knutson, T. Tao:)

If α, β, γ are partitions, then $c^{\gamma}_{\alpha,\beta} > 0$ iff.

- $\sum_i \gamma_i = \sum_i \alpha_i + \sum_i \beta_i$, *and*
- *the inequalities*

$$\sum_{k \in K} \gamma_k \leqslant \sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j \qquad (*_{IJK})$$

holds for all (I, J, K) in T_r for all $r < n$.

Connections with LR Coefficients

At the level of combinatorics, the connection is:

Theorem (A. Knutson, T. Tao:)

If α, β, γ are partitions, then $c_{\alpha,\beta}^{\gamma} > 0$ iff.

- $\sum_i \gamma_i = \sum_i \alpha_i + \sum_i \beta_i$, *and*
- *the inequalities*

$$\sum_{k \in K} \gamma_k \leqslant \sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j \qquad (*_{IJK})$$

holds for all (I, J, K) in T_r for all $r < n$.

The problem of Hermitian matrices can be related to Schubert calculus. I'll try to explain this later.

Table of Contents

1 Defining LR Coefficients

2 Decomposing Representations of $\mathbf{GL}_n$

3 Eigenvalues of Hermitian Matrices

4 Extensions of Finite Abelian p-Groups

5 Intersection of Schubert Cells

6 Underlying Connections

An Exercise from Hatcher

Exercise 2.1.14 from A. Hatcher's *Algebraic Topology*:

An Exercise from Hatcher

Exercise 2.1.14 from A. Hatcher's *Algebraic Topology*:

Exercise

Determine whether there exists a short exact sequence

$$0 \longrightarrow \mathbb{Z}/4 \longrightarrow \mathbb{Z}/8 \oplus \mathbb{Z}/2 \longrightarrow \mathbb{Z}/4 \longrightarrow 0.$$

An Exercise from Hatcher

Exercise 2.1.14 from A. Hatcher's *Algebraic Topology*:

Exercise

Determine whether there exists a short exact sequence

$$0 \longrightarrow \mathbb{Z}/4 \longrightarrow \mathbb{Z}/8 \oplus \mathbb{Z}/2 \longrightarrow \mathbb{Z}/4 \longrightarrow 0.$$

Let $\mathbb{Z}/4 \hookrightarrow \mathbb{Z}/8 \oplus \mathbb{Z}/2$ be defined by $1 \mapsto (2,1)$.

An Exercise from Hatcher

Exericse (continued)

More generally, determine which abelian groups A fit into a short exact sequence

$$0 \longrightarrow \mathbb{Z}/p^m \longrightarrow A \longrightarrow \mathbb{Z}/p^n \longrightarrow 0.$$

An Exercise from Hatcher

Exericse (continued)

More generally, determine which abelian groups A fit into a short exact sequence

$$0 \longrightarrow \mathbb{Z}/p^m \longrightarrow A \longrightarrow \mathbb{Z}/p^n \longrightarrow 0.$$

Some homological algebra $\implies$ $A = \mathbb{Z}/p^{m+n}$ or $\mathbb{Z}/p^a \oplus \mathbb{Z}/p^{m+n-b}$, where $a \leqslant \min\{m, n\}$.

An Exercise from Hatcher

Exericse (continued)

More generally, determine which abelian groups A fit into a short exact sequence

$$0 \longrightarrow \mathbb{Z}/p^m \longrightarrow A \longrightarrow \mathbb{Z}/p^n \longrightarrow 0.$$

Some homological algebra $\implies$ $A = \mathbb{Z}/p^{m+n}$ or $\mathbb{Z}/p^a \oplus \mathbb{Z}/p^{m+n-b}$, where $a \leqslant \min\{m, n\}$.
A is called an **extension** of $\mathbb{Z}/p^n$ by $\mathbb{Z}/p^m$.

Extension Problem

A more general question is:

Question

Given finite abelian p-groups $\bigoplus_i \mathbb{Z}/p^{\alpha_i}$, $\bigoplus_i \mathbb{Z}/p^{\beta_i}$, which abelian group A can be the extension of them? In other words, for which A can we have a short exact sequence

$$0 \longrightarrow \bigoplus_i \mathbb{Z}/p^{\alpha_i} \longrightarrow A \longrightarrow \bigoplus_i \mathbb{Z}/p^{\beta_i} \longrightarrow 0 \ ?$$

Extension Problem

A more general question is:

Question

Given finite abelian p-groups $\bigoplus_i \mathbb{Z}/p^{\alpha_i}$, $\bigoplus_i \mathbb{Z}/p^{\beta_i}$, which abelian group A can be the extension of them? In other words, for which A can we have a short exact sequence

$$0 \longrightarrow \bigoplus_i \mathbb{Z}/p^{\alpha_i} \longrightarrow A \longrightarrow \bigoplus_i \mathbb{Z}/p^{\beta_i} \longrightarrow 0 \text{ ?}$$

(One can easily compute $\mathrm{Ext}^1_{\mathbb{Z}}(\bigoplus_i \mathbb{Z}/p^{\alpha_i}, \bigoplus_i \mathbb{Z}/p^{\beta_i})$, which classifies all abelian group extensions. But this doesn't tell us what those extensions are.)

Observations

Let's make some first step observations.

$$0 \longrightarrow \bigoplus_i \mathbb{Z}/p^{\alpha_i} \longrightarrow A \longrightarrow \bigoplus_i \mathbb{Z}/p^{\beta_i} \longrightarrow 0$$

Observations

Let's make some first step observations.

$$0 \longrightarrow \bigoplus_i \mathbb{Z}/p^{\alpha_i} \longrightarrow A \longrightarrow \bigoplus_i \mathbb{Z}/p^{\beta_i} \longrightarrow 0$$

- By counting elements, we know A is necessarily a p-group. So $A = \bigoplus_i \mathbb{Z}/p^{\gamma_i}$ for some γ.

Observations

Let's make some first step observations.

$$0 \longrightarrow \bigoplus_i \mathbb{Z}/p^{\alpha_i} \longrightarrow A \longrightarrow \bigoplus_i \mathbb{Z}/p^{\beta_i} \longrightarrow 0$$

- By counting elements, we know A is necessarily a p-group. So $A = \bigoplus_i \mathbb{Z}/p^{\gamma_i}$ for some γ.
- Any finite abelian p-group determines, and is determined by a partition:

$$M_\gamma := \bigoplus_i \mathbb{Z}/p^{\gamma_i} \leftrightsquigarrow \gamma = (\gamma_1 \geqslant \gamma_2 \geqslant \cdots).$$

γ is called the **type** of M_γ.

Observation 1

Let $G^{\gamma}_{\alpha,\beta}(p)$ denote the number of non-isomorphic extensions.

$$0 \longrightarrow M_\alpha \longrightarrow M_\gamma \longrightarrow M_\beta \longrightarrow 0$$

Observation 1

Let $G^{\gamma}_{\alpha,\beta}(p)$ denote the number of non-isomorphic extensions.

$$0 \longrightarrow M_\alpha \longrightarrow M_\gamma \longrightarrow M_\beta \longrightarrow 0$$

- The trivial observation: $M_\alpha \subseteq M_\gamma$.

Observation 1

Let $G^{\gamma}_{\alpha,\beta}(p)$ denote the number of non-isomorphic extensions.

$$0 \longrightarrow M_\alpha \longrightarrow M_\gamma \longrightarrow M_\beta \longrightarrow 0$$

- The trivial observation: $M_\alpha \subseteq M_\gamma$.
  $G^{\gamma}_{\alpha,\beta}(p) > 0 \implies \alpha \subseteq \gamma$.

Observation 2

$$0 \longrightarrow M_\alpha \longrightarrow M_\gamma \longrightarrow M_\beta \longrightarrow 0$$

Observation 2

$$0 \longrightarrow M_\alpha \longrightarrow M_\gamma \longrightarrow M_\beta \longrightarrow 0$$

- The length of these groups must add up: $\ell(M_\gamma) = \ell(M_\alpha) + \ell(M_\beta)$.

Observation 2

$$0 \longrightarrow M_\alpha \longrightarrow M_\gamma \longrightarrow M_\beta \longrightarrow 0$$

- The length of these groups must add up:
  $\ell(M_\gamma) = \ell(M_\alpha) + \ell(M_\beta)$.
  One way to compute length:

$$\ell(M) = \sum_{i \geqslant 1} \dim_{\mathbb{Z}/p}(p^{i-1}M/p^iM).$$

Observation 2

$$0 \longrightarrow M_\alpha \longrightarrow M_\gamma \longrightarrow M_\beta \longrightarrow 0$$

- The length of these groups must add up: $\ell(M_\gamma) = \ell(M_\alpha) + \ell(M_\beta)$.
  One way to compute length:
  $$\ell(M) = \sum_{i \geqslant 1} \dim_{\mathbb{Z}/p}(p^{i-1}M/p^iM).$$
  In fact, $\dim_{\mathbb{Z}/p}(p^{i-1}M_\alpha/p^iM_\alpha) = \alpha_i^T$.

Observation 2

$$0 \longrightarrow M_\alpha \longrightarrow M_\gamma \longrightarrow M_\beta \longrightarrow 0$$

- The length of these groups must add up:
  $\ell(M_\gamma) = \ell(M_\alpha) + \ell(M_\beta)$.
  One way to compute length:

$$\ell(M) = \sum_{i \geqslant 1} \dim_{\mathbb{Z}/p}(p^{i-1}M/p^i M).$$

  In fact, $\dim_{\mathbb{Z}/p}(p^{i-1}M_\alpha/p^i M_\alpha) = \alpha_i^T$.
  $G^\gamma_{\alpha,\beta}(p) > 0 \implies |\gamma^T| = |\alpha^T| + |\beta^T| \implies |\gamma| = |\alpha| + |\beta|$.

Observations

These conditions is the same as $\alpha \subseteq \gamma$ and $|\gamma \backslash \alpha| = |\beta|$, a necessary condition for $c^{\gamma}_{\alpha,\beta}$ to be nonzero.

Relation to LR Coefficients

By manipulating lengths (which involves long combinatorial calculation) and some group theory (e.g. Pontryagin duality), one can show:

Theorem

Given an extension $0 \to M_\alpha \to M_\gamma \to M_\beta \to 0$*, for each* $i \geqslant 0$*, let* $\gamma^{(i)}$ *be the type of* $M_\gamma / p^i M_\alpha$*. Then the sequence*

$$\gamma^{(0)T} \subseteq \gamma^{(1)T} \subseteq \cdots \subseteq \gamma^{(r)T}$$

(where $p^r M_\alpha = 0$*) corresponds to an LR tableau for the triple* (α, β, γ)*.*

Relation to LR Coefficients

Relation to LR Coefficients

- Think of $\gamma^{(0)T} \subseteq \gamma^{(1)T} \subseteq \cdots \subseteq \gamma^{(r)T}$ as Young diagrams contained successively into the next.

Relation to LR Coefficients

- Think of $\gamma^{(0)T} \subseteq \gamma^{(1)T} \subseteq \cdots \subseteq \gamma^{(r)T}$ as Young diagrams contained successively into the next.
- Each difference $\gamma^{(i)T} \backslash \gamma^{(i-1)T}$ has only one box at each column.

Relation to LR Coefficients

- Think of $\gamma^{(0)T} \subseteq \gamma^{(1)T} \subseteq \cdots \subseteq \gamma^{(r)T}$ as Young diagrams contained successively into the next.
- Each difference $\gamma^{(i)T} \backslash \gamma^{(i-1)T}$ has only one box at each column.
  $\implies$ filling in i in $\gamma^{(i)T} \backslash \gamma^{(i-1)T}$ gives us a tableau whose rows are weakly increasing, columns are strongly increasing.

As a result

Corollary

$(c^{\gamma^T}_{\alpha^T,\beta^T} =) c^{\gamma}_{\alpha,\beta} = 0 \implies G^{\gamma}_{\alpha,\beta} = 0.$

In fact, more is true:

Theorem

For any triple α, β, γ, $\exists\, ! g^{\gamma}_{\alpha,\beta} \in \mathbb{Z}[t]$, such that $G^{\gamma}_{\alpha,\beta}(p) = g^{\gamma}_{\alpha,\beta}(p)$. In fact, if $c^{\gamma}_{\alpha,\beta} = 0$ then $g^{\gamma}_{\alpha,\beta} = 0$; otherwise,

$$g^{\gamma}_{\alpha,\beta}(t) = c^{\gamma}_{\alpha,\beta}(t-1)^N + a_{N-1}(t-1)^{N-1} + \cdots + a_0$$

for some $a_i \in \mathbb{Z}_{\geqslant 0}$.

As a result

Corollary

$(c_{\alpha^T,\beta^T}^{\gamma^T} =) c_{\alpha,\beta}^{\gamma} = 0 \implies G_{\alpha,\beta}^{\gamma} = 0.$

In fact, more is true:

Theorem

For any triple α, β, γ, $\exists ! g_{\alpha,\beta}^{\gamma} \in \mathbb{Z}[t]$, *such that* $G_{\alpha,\beta}^{\gamma}(p) = g_{\alpha,\beta}^{\gamma}(p)$. *In fact, if* $c_{\alpha,\beta}^{\gamma} = 0$ *then* $g_{\alpha,\beta}^{\gamma} = 0$; *otherwise,*

$$g_{\alpha,\beta}^{\gamma}(t) = c_{\alpha,\beta}^{\gamma}(t-1)^N + a_{N-1}(t-1)^{N-1} + \cdots + a_0$$

for some $a_i \in \mathbb{Z}_{\geqslant 0}$.

Therefore $c_{\alpha,\beta}^{\gamma} > 0 \iff G_{\alpha,\beta}^{\gamma}(p) > 0$ for some p $\iff G_{\alpha,\beta}^{\gamma}(p) > 0$ for all p.

Generalizations

This theorem is still true if we instead consider finite length modules over a discrete valuation ring.

Table of Contents

1 Defining LR Coefficients

2 Decomposing Representations of $\mathbf{GL}_n$

3 Eigenvalues of Hermitian Matrices

4 Extensions of Finite Abelian p-Groups

5 Intersection of Schubert Cells

6 Underlying Connections

Grassmannians

Definition (Grassmannian)

Let V be an n-dimensional vector space (say over $\mathbb{C}$), $0 \leqslant r \leqslant n$. The Grassmannian $\mathbf{Gr}_r(V)$ is the set of r-dimensional subspaces of V.

Grassmannians

Definition (Grassmannian)

Let V be an n-dimensional vector space (say over $\mathbb{C}$), $0 \leqslant r \leqslant n$. The Grassmannian $\mathbf{Gr}_r(V)$ is the set of r-dimensional subspaces of V.

There is a way to give this an analytic structure. With this structure $\mathbf{Gr}_r(V)$ is a compact complex manifold of dimension $r(n-r)$.

Schubert Decomposition

There are some nice subspaces in $\mathbf{Gr}_r(V)$. Fix a basis $E = \{e_1, \dots, e_n\}$ of V. For any strictly increasing sequence $P = (1 \leqslant p_1 < p_2 < \cdots < p_r \leqslant n)$, define the **Schubert cell** corresponding to P to be

$$X(E)_P := \{L \in \mathbf{Gr}_r(X) \mid \forall\, i, \dim(L \cap \operatorname{span}\{e_1, \dots, e_i\}) \geqslant i\}.$$

In general these subspaces have singularities.

Schubert Decomposition

There are some nice subspaces in $\mathbf{Gr}_r(V)$. Fix a basis $E = \{e_1, \ldots, e_n\}$ of V. For any strictly increasing sequence $P = (1 \leqslant p_1 < p_2 < \cdots < p_r \leqslant n)$, define the **Schubert cell** corresponding to P to be

$$X(E)_P := \{L \in \mathbf{Gr}_r(X) \mid \forall\, i, \dim(L \cap \operatorname{span}\{e_1, \ldots, e_i\}) \geqslant i\}.$$

In general these subspaces have singularities.
As before, we want to parametrize these objects using partitions.
Letting $\lambda_j := n - r + j - p_j$, $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r)$ is a partition. Let $X(E)_\lambda := X(E)_P$.

Schubert Decomposition

These subspaces are nice because of the following. $\mathbf{GL}(V) \circlearrowright V$, so $B = \{\text{upper-}\Delta \text{ matrices}\} \subset \mathbf{GL}(V) \circlearrowright \mathbf{Gr}_r(V)$, and $\mathbf{Gr}_r(V)$ is a disjoint union of orbits under the action of B.

Theorem

Schubert Decomposition For each λ, there is an orbit $X(E)^\circ_\lambda$ isomorphic to $\mathbb{C}^{r(n-r)-|\lambda|}$ such that $\overline{X(E)^\circ_\lambda} = X(E)_\lambda$.
Also, $\mathbf{Gr}_r(V) = \coprod_{\lambda \subseteq r\times(n-r)} X^\circ_\lambda$.

Schubert Decomposition

These subspaces are nice because of the following. $\mathbf{GL}(V) \circlearrowright V$, so $B = \{\text{upper-}\Delta \text{ matrices}\} \subset \mathbf{GL}(V) \circlearrowright \mathbf{Gr}_r(V)$, and $\mathbf{Gr}_r(V)$ is a disjoint union of orbits under the action of B.

Theorem

Schubert Decomposition For each λ, there is an orbit $X(E)^\circ_\lambda$ isomorphic to $\mathbb{C}^{r(n-r)-|\lambda|}$ such that $\overline{X(E)^\circ_\lambda} = X(E)_\lambda$.
Also, $\mathbf{Gr}_r(V) = \coprod_{\lambda \subseteq r\times(n-r)} X^\circ_\lambda$.

In topological point of view, this gives us a CW structure on $\mathbf{Gr}_r(V)$, and $X(E)_\lambda$ is a $2(r(n-r)-|\lambda|)$-cell. One can compute cohomologies using this decomposition.

Intersections

People want to understand the intersection between the cells. A basic result is:

Proposition

$X(E)_\lambda = \coprod_{\mu \subseteq \lambda} X(E)^\circ_\mu = \bigcup_{\mu \subseteq \lambda} X(E)_\mu.$

Intersections

People want to understand the intersection between the cells. A basic result is:

Proposition

$X(E)_\lambda = \coprod_{\mu \subseteq \lambda} X(E)^\circ_\mu = \bigcup_{\mu \subseteq \lambda} X(E)_\mu$.

Therefore the intersection of two Schubert cells will be a union of smaller cells. So intersections are easier to compute.

Intersections

People want to understand the intersection between the cells. A basic result is:

Proposition

$X(E)_\lambda = \coprod_{\mu\subseteq\lambda} X(E)^\circ_\mu = \bigcup_{\mu\subseteq\lambda} X(E)_\mu.$

Therefore the intersection of two Schubert cells will be a union of smaller cells. So intersections are easier to compute.
What we want to understand is when the two cells $X(E)_\lambda$ and $X(F)_\mu$ are defined using different bases $E \neq F$.

Intersections

People want to understand the intersection between the cells. A basic result is:

Proposition

$X(E)_\lambda = \coprod_{\mu \subseteq \lambda} X(E)^\circ_\mu = \bigcup_{\mu \subseteq \lambda} X(E)_\mu.$

Therefore the intersection of two Schubert cells will be a union of smaller cells. So intersections are easier to compute.

Question

Given two distinct bases E, F of V and two partitions $\lambda, \mu \subseteq r \times (n-r)$ of length r, what's the intersection $X(E)_\lambda \cap X(F)_\mu$?

Cohomology

To study this question, we need to translate it to something we can work on.

Cohomology

To study this question, we need to translate it to something we can work on.
Remember that each $X(E)_\lambda$ is a $2(r(n-r)-|\lambda|)$-cell in $\mathbf{Gr}_r(V)$.
Therefore it gives a class $[X_\lambda]$ in $H_{2(r(n-r)-|\lambda|)}(\mathbf{Gr}_r(V)) \cong H^{2|\lambda|}(\mathbf{Gr}_r(V))$. This class doesn't depend on the choice of basis E.

Cohomology

To study this question, we need to translate it to something we can work on.
Remember that each $X(E)_\lambda$ is a $2(r(n-r)-|\lambda|)$-cell in $\mathbf{Gr}_r(V)$.
Therefore it gives a class $[X_\lambda]$ in $H_{2(r(n-r)-|\lambda|)}(\mathbf{Gr}_r(V)) \cong H^{2|\lambda|}(\mathbf{Gr}_r(V))$. This class doesn't depend on the choice of basis E.
Things are easier to compute here, because $\bigoplus_i H^i(\mathbf{Gr}_r(V))$ is not just a group, but also a ring (under the cup product).

Cohomology

To study this question, we need to translate it to something we can work on.
Remember that each $X(E)_\lambda$ is a $2(r(n-r)-|\lambda|)$-cell in $\mathbf{Gr}_r(V)$.
Therefore it gives a class $[X_\lambda]$ in $H_{2(r(n-r)-|\lambda|)}(\mathbf{Gr}_r(V)) \cong H^{2|\lambda|}(\mathbf{Gr}_r(V))$. This class doesn't depend on the choice of basis E.
Things are easier to compute here, because $\bigoplus_i H^i(\mathbf{Gr}_r(V))$ is not just a group, but also a ring (under the cup product).

Theorem

If E and F are different bases of V, then

$$[X_\lambda] \smile [X_\mu] = [X(E)_\lambda \cap X(F)_\mu].$$

Connection with LR Coefficients

Theorem

$$[X_\alpha] \smile [X_\beta] = \sum_{\gamma \subseteq r \times (n-r)} c^{\gamma}_{\alpha,\beta} [X_\gamma].$$

Connection with LR Coefficients

Theorem

$$[X_\alpha] \smile [X_\beta] = \sum_{\gamma \subseteq r \times (n-r)} c^{\gamma}_{\alpha,\beta} [X_\gamma].$$

What this means is that, for generic choice of distinct bases E, F, G of V, the intersection

$$X(E)_\alpha \cap X(F)_\beta \cap X(G)_{\gamma^\vee}$$

consists of $c^{\gamma}_{\alpha,\beta}$ many points.

Table of Contents

1 Defining LR Coefficients

2 Decomposing Representations of $\mathbf{GL}_n$

3 Eigenvalues of Hermitian Matrices

4 Extensions of Finite Abelian p-Groups

5 Intersection of Schubert Cells

6 Underlying Connections

Connections

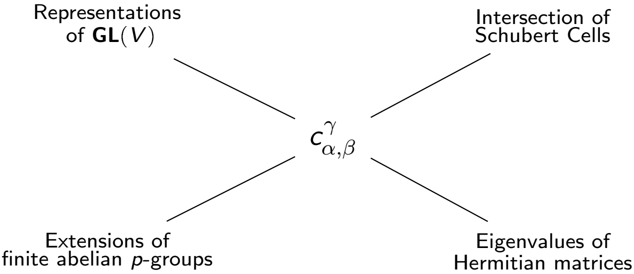
Representations of $\mathbf{GL}(V)$

Intersection of Schubert Cells

Extensions of finite abelian p-groups

Eigenvalues of Hermitian matrices

Connections



Reps of $\mathbf{GL}(V) \leftrightsquigarrow$ Schubert Cells

A connection between these two areas is via the proof of the theorem

Theorem

$$[X_\alpha] \smile [X_\beta] = \sum_{\gamma \subseteq r \times (n-r)} c^{\gamma}_{\alpha,\beta} [X_\gamma].$$

It goes as follows:

Proof of the theorem

- For each $S_\lambda V$, we can get a polynomial:

$$s_\lambda = \operatorname{Tr} \rho_\lambda \operatorname{diag}(x_1, \dots, x_n)$$

(where any $A \in \mathbf{GL}(V)$ acts on $S_\lambda V$ by the matrix $\rho_\lambda A$), called the **character** of $S_\lambda V$.

Proof of the theorem

- For each $S_\lambda V$, we can get a polynomial:

$$s_\lambda = \operatorname{Tr} \rho_\lambda \operatorname{diag}(x_1, \ldots, x_n)$$

(where any $A \in \mathbf{GL}(V)$ acts on $S_\lambda V$ by the matrix $\rho_\lambda A$), called the **character** of $S_\lambda V$.

-

$$S_\lambda V \otimes S_\mu V = \bigoplus_\nu (S_\nu V)^{\oplus c^\nu_{\lambda,\mu}} \rightsquigarrow s_\lambda \cdot s_\mu = \sum_\nu c^\nu_{\lambda,\mu} s_\nu .$$

Proof of the theorem

- For each $S_\lambda V$, we can get a polynomial:

$$s_\lambda = \operatorname{Tr} \rho_\lambda \operatorname{diag}(x_1, \ldots, x_n)$$

  (where any $A \in \mathbf{GL}(V)$ acts on $S_\lambda V$ by the matrix $\rho_\lambda A$), called the **character** of $S_\lambda V$.

- $$S_\lambda V \otimes S_\mu V = \bigoplus_\nu (S_\nu V)^{\oplus c^\nu_{\lambda,\mu}} \rightsquigarrow s_\lambda \cdot s_\mu = \sum_\nu c^\nu_{\lambda,\mu} s_\nu .$$

- $\{s_\lambda\}_\lambda$ form a basis of the ring Λ of all symmetric polynomials with integer coefficients (i.e. $\Lambda = \mathbb{C}[x_1, \ldots, x_n]^{S_n}$).

Proof of the theorem

- Show that $\{[X_\lambda]\}_\lambda$ satisfies the same product rule as $\{s_\lambda\}_\lambda$ does.

Proof of the theorem

- Show that $\{[X_\lambda]\}_\lambda$ satisfies the same product rule as $\{s_\lambda\}_\lambda$ does.
- Show that $\Lambda \to H^*(\mathbf{Gr}_r(V))$, $s_\lambda \mapsto [X_\lambda]$ is a ring isomorphism.

Proof of the theorem

- Show that $\{[X_\lambda]\}_\lambda$ satisfies the same product rule as $\{s_\lambda\}_\lambda$ does.
- Show that $\Lambda \to H^*(\mathbf{Gr}_r(V))$, $s_\lambda \mapsto [X_\lambda]$ is a ring isomorphism.

$$\mathrm{PolyRep}(\mathbf{GL}(V)) \leftrightsquigarrow \Lambda \leftrightsquigarrow H^*(\mathbf{Gr}_r(V)).$$

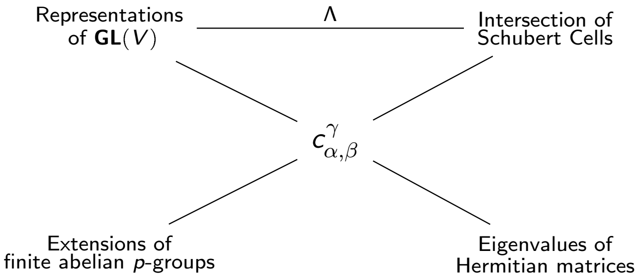
Reps of $\mathbf{GL}(V) \leftrightsquigarrow$ Schubert Cells

One would expect a connection without going through the combinatorics, since we have a natural action $B \subseteq \mathbf{GL}(V) \circlearrowright \mathbf{Gr}_r(V)$.

Reps of $\mathbf{GL}(V) \leftrightsquigarrow$ Schubert Cells

One would expect a connection without going through the combinatorics, since we have a natural action
$B \subseteq \mathbf{GL}(V) \circlearrowright \mathbf{Gr}_r(V)$.
But such a connection (that's at the same time conceptually satisfying) will use some higher machineries. There isn't a easy way to describe this.

Connections



Schubert Cells $\leftrightsquigarrow$ Hermitian Matrices

The key to relating these two areas is **Rayleigh trace**. If A is an $n \times n$ Hermitian matrix, then for any subspace $L \subseteq \mathbb{C}^n$, the Rayleigh trace is defined to be

$$R_A(L) := \mathrm{Tr}(L \hookrightarrow \mathbb{C}^n \xrightarrow{A} \mathbb{C}^n \xrightarrow{\mathrm{pr}} L).$$

Schubert Cells $\leftrightsquigarrow$ Hermitian Matrices

The key to relating these two areas is **Rayleigh trace**. If A is an $n \times n$ Hermitian matrix, then for any subspace $L \subseteq \mathbb{C}^n$, the Rayleigh trace is defined to be

$$R_A(L) := \mathrm{Tr}(L \hookrightarrow \mathbb{C}^n \xrightarrow{A} \mathbb{C}^n \xrightarrow{\mathrm{pr}} L).$$

Let $\alpha = (\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_n)$ be eigenvalues of A and $v_1, \ldots, v_n$ the corresponding eigenvectors. The v_i's form an ordered basis $E(A)$ of $\mathbb{C}^n$.

Schubert Cells $\leftrightsquigarrow$ Hermitian Matrices

Theorem (J. Hersch and B. Zwahlen)

For any subset $P = \{p_1 < \cdots < p_r\} \subseteq \{1, \ldots, n\}$,

$$\sum_{i \in I} \alpha_i = \min_{L \in X(E(A))_P} R_A(L).$$

Schubert Cells $\leftrightsquigarrow$ Hermitian Matrices

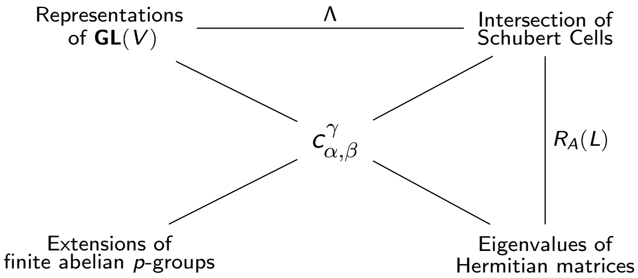
Theorem (J. Hersch and B. Zwahlen)

For any subset $P = \{p_1 < \cdots < p_r\} \subseteq \{1, \ldots, n\}$,

$$\sum_{i \in I} \alpha_i = \min_{L \in X(E(A))_P} R_A(L).$$

Using this result one can go back and forth between the two areas.

Connections



Measure of Multiplicities

The LR coefficients measure some kind of “multiplicity” in Rep theory, Schubert calculus and p-group extensions problems. Is there a similar interpretation in Hermitian eigenvalue problems?

Measure of Multiplicities

The LR coefficients measure some kind of "multiplicity" in Rep theory, Schubert calculus and p-group extensions problems. Is there a similar interpretation in Hermitian eigenvalue problems? Knutson: $c^{\gamma}_{\alpha,\beta}$ is the asymptotic "volume" of $\{(A, B, C = A + B) \mid A, B, C$ have respective eigenvalues $N\alpha, N\beta, N\gamma\}$ as $N \to \infty$.

The End