Your favorite surprising connections in Mathematics

There are certain things in mathematics that have caused me a pleasant surprise -- when some part of mathematics is brought to bear in a fundamental way on another, where the connection between the two is unexpected. The first example that comes to my mind is the proof by Furstenberg and Katznelson of Szemerédi's theorem on the existence of arbitrarily long arithmetic progressions in a set of integers which has positive upper Banach density, but using ergodic theory. Of course in the years since then, this idea has now become enshrined and may no longer be viewed as surprising, but it certainly was when it was first devised.

Another unexpected connection was when Kolmogorov used Shannon's notion of probabilistic entropy as an important invariant in dynamical systems.

So, what other surprising connections are there out there?
The ubiquity of Littlewood-Richardson coefficients. Given three partitions \( \lambda, \mu, \nu \) each with at most \( n \) parts, there is a combinatorial definition for a number \( c_{\lambda, \mu}^{\nu} \) which is nonzero if and only if any of the following statements are true:

- There exist Hermitian matrices \( A, B, C \) whose eigenvalues are \( \lambda, \mu, \nu \), respectively and \( A + B = C \) (one can also replace Hermitian by real symmetric).
- The irreducible representation of \( \text{GL}_n(\mathbb{C}) \) with highest weight \( \nu \) is a subrepresentation of the tensor product of those irreducible representations with highest weights \( \lambda \) and \( \mu \).
- Indexing the Schubert cells of the Grassmannian \( \text{Gr}(d, \mathbb{C}^m) \) (where \( d \geq n \) and \( m - d \) is at least as big as any part of \( \lambda, \mu, \nu \)) by \( \sigma_\lambda \) appropriately, the cycle \( \sigma_\nu \) appears in the intersection product \( \sigma_\lambda \sigma_\mu \).
- There exists finite Abelian \( p \)-groups \( A, B, C \) and a short exact sequence
  \[
  0 \to A \to B \to C \to 0
  \]
  such that \( B \cong \bigoplus_i \mathbb{Z}/p^{k_i} \), \( A \cong \bigoplus_i \mathbb{Z}/p^{\lambda_i} \), and \( C \cong \bigoplus_i \mathbb{Z}/p^{\mu_i} \).

And probably many more things.
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6. Underlying Connections
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Combinatorial Definition of LR Coefficients

Definition (Partitions and Young diagrams)

A partition of $n \in \mathbb{N}_+$ is a weakly decreasing integer sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ that sums up to $n$, i.e. $\sum_i \lambda_i = n$. Denoted $\lambda \vdash n$. Define $|\lambda| = \sum_i \lambda_i$.

A Young diagram is a collection of finitely many “top-left aligned” boxes.

{Paritions of $n$} $\leftrightarrow$ {Young diagrams with $n$ boxes}. 
Combinatorial Definition of LR Coefficients

\[ \lambda = (5, 3, 2) \vdash 10 \]

\[ \mu = \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} \subset \begin{array}{c}
\end{array} = \lambda \rightsquigarrow \lambda \backslash \mu = \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} \]
Combinatorial Definition of LR Coefficients

We can also fill in numbers

\[
\begin{array}{cccc}
1 & 6 & 9 & 4 & 3 \\
9 & 8 & 7 \\
6 & 1 \\
\end{array}
\]
Combinatorial Definition of LR Coefficients

Let \( \alpha, \beta, \gamma \) be Young diagrams (or partitions).
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Let $\alpha, \beta, \gamma$ be Young diagrams (or partitions).

Definition (Littlewood-Richardson Coefficients)

The LR coefficients $c_{\alpha, \beta}^\gamma$ is defined as follows.

- If $\alpha \subseteq \gamma$ and $|\gamma \setminus \alpha| = |\beta|$, then $c_{\alpha, \beta}^\gamma$ is the number of ways to fill numbers into $\gamma \setminus \alpha$ such that:
  
  1. each row is weakly increasing, each column is strongly increasing;
  2. the number $i$ appears $\beta_i$ times; and
  3. if we concatenate the rows of $\gamma \setminus \alpha$ (start from the bottom row, end at top row), read the numbers from right to left, then a larger number should not appear less often or as often than a smaller number (i.e. a “lattice word”).

(A filled diagram following these rules is called a LR tableau.)
Combinatorial Definition of LR Coefficients

Definition (Littlewood-Richardson Coefficients)

- Otherwise set $c^\gamma_{\alpha,\beta} = 0$. 
Example

\[ \alpha = \begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array}, \quad \beta = (3, 3, 1), \quad \gamma = \begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array}. \]
Example

\[ \alpha = \begin{array}{c}
\hline
& & \\
\end{array}, \quad \beta = (3, 3, 1), \quad \gamma = \begin{array}{c}
\hline
& & & \\
& & & \\
\end{array}. \]

Fill \{1, 1, 1, 2, 2, 2, 3\} in \( \gamma \setminus \alpha = \begin{array}{c}
\hline
& & & \\
& & & \\
\end{array}, \)
Example

\[
\begin{array}{ccc}
1 & 1 & 1 \\
2 & 2 & \\
2 & 3 & \\
\end{array}
\]

is an LR tableau:
Example

\[
\begin{array}{ccc}
1 & 1 & 1 \\
2 & 2 & \\
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\end{array}
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✓ each row is weakly increasing, each column is strongly increasing;
Example

is an LR tableau:

✓ each row is weakly increasing, each column is strongly increasing;
✓ the number $i$ appears $\beta_i$ times ($\beta = (3, 3, 1)$;
Ubiquity of Littlewood-Richardson Coefficients

Defining LR Coefficients

Read from right to left:
Ubiquity of Littlewood-Richardson Coefficients

Defining LR Coefficients
Read from right to left:

1
11
111
2111
22111
322111
2322111
Example

\( \alpha = (2, 1), \beta = (3, 3, 1), \gamma = (5, 3, 2) \).
It’s easy to see that

\[
\begin{array}{ccc}
1 & 1 & 1 \\
2 & 2 \\
2 & 3 \\
\end{array}
\]

is the only LR tableau of our triple \((\alpha, \beta, \gamma)\). Therefore \(c_{\alpha, \beta}^{\gamma} = 1\).
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Historically the LR coefficients are not defined combinatorially, but actually in representation theoretic context. Littlewood and Richardson proved the combinatorial characterization of these coefficients.
A representation of a group $G$ is a (finite dimensional) vector space $V$ on which $G$ acts by linear automorphisms.
A representation of a group $G$ is a (finite dimensional) vector space $V$ on which $G$ acts by linear automorphisms. For example, any vector space $V$ is a representation of $\text{GL}(V)$.
Examples of Representations

We’ve seen $V$ is a representation of $\text{GL}(V)$. It’s irreducible.
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$$A \cdot (v_1 \otimes \cdots \otimes v_m) = (Av_1) \otimes \cdots \otimes (Av_m).$$

Not irreducible
Examples of Representations

We’ve seen $V$ is a representation of $\text{GL}(V)$. It’s irreducible. $V^\otimes d$ is also a representation of $\text{GL}(V)$:

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Not irreducible

$\text{Sym}^d V = \mathbb{C}[e_1, \ldots, e_{\dim V}]_d, \wedge^d V$, both irreducible
Irreducible Representations of $\text{GL}(V)$

$$\text{Sym}^5 V \leftrightarrow \begin{array}{cccc}
\vdots \\
\vdots \\
\vdots \\
\end{array}$$

$$\bigwedge^5 V \leftrightarrow \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\end{array}.$$
Irreducible Representations of $\text{GL}(V)$

\begin{align*}
\begin{array}{c}
\begin{bmatrix}
1 & 1 \\
1 & 1 \\
1 & 1 \\
1 & 1 \\
\end{bmatrix}
\end{array}
& \longmapsto "(\text{Sym}^3 V) \wedge (\text{Sym}^2 V)"? \\
\begin{bmatrix}
2 & 2 & 1 \\
\end{bmatrix}
& \longmapsto "\left(\bigwedge^2 V\right) \text{Sym}\left(\bigwedge^2 V\right) \text{Sym}\left(\bigwedge^1 V\right)"?
\end{align*}
Irreducible Representations of $\mathbf{GL}(V)$

There's a correct way to jazz up $\text{Sym}$ and $\bigwedge$.

\[ \begin{array}{ccc}
\text{ } & & \\
\text{ } & & \\
\text{ } & & \\
\end{array} \quad \leftrightarrow \quad S_{(3,2)} V. \]

\[ \lambda \leftrightarrow S_{\lambda} V. \]

Called \textit{Schur modules}. 
Irreducible Representations of $\text{GL}(V)$

Theorem

Irreducible polynomial representations of $\text{GL}(V)$ are exactly the $S_\lambda V$’s, as $\lambda$ ranges over partitions of length $\leq \dim V$ (i.e. $\lambda = (\lambda_1 \geq \cdots \geq \lambda_{\dim V} \geq 0 \geq \cdots)$).
LR Coefficients in Representations of $\text{GL}(V)$

We can tensor two irreps. This will in general result in a reducible representation.
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$$S_\alpha V \otimes_{\mathbb{C}} S_\beta V = \bigoplus_{\gamma} (S_\gamma V) \oplus d_{\alpha,\beta}^\gamma$$
We can tensor two irreps. This will in general result in a reducible representation.

\[ S_\alpha V \otimes_C S_\beta V = \bigoplus_{\gamma} (S_\gamma V)^{\oplus d_{\gamma, \alpha, \beta}} \]

**Theorem**

The multiplicities \( d_{\gamma, \alpha, \beta} \) equals the LR coefficients \( c_{\alpha, \beta}^{\gamma} \). In other words, \( c_{\alpha, \beta}^{\gamma} \) is exactly the number of times \( S_\gamma V \) appears in \( S_\alpha V \otimes S_\beta V \).
LR Coefficients in Representations of $\text{GL}(V)$

Using this, we can go back and forth between results on $c^\gamma_{\alpha,\beta}$ and results about the $S_\lambda V$’s.
LR Coefficients in Representations of $\text{GL}(V)$

Using this, we can go back and forth between results on $c_{\alpha,\beta}^\gamma$ and results about the $S_\lambda V$’s. For example, because $S_\alpha V \otimes S_\beta V \cong S_\beta V \otimes S_\alpha V$, we know $c_{\alpha,\beta}^\gamma = c_{\beta,\alpha}^\gamma$. This is not obvious from the definition of $c_{\alpha,\beta}^\gamma$. 
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**Question**

Given descending sequences $\alpha, \beta, \gamma$ of length $n$, are there Hermitian matrices $A, B, C$ with eigenvalues $\alpha, \beta, \gamma$ so that $A + B = C$?
Necessary Conditions

- Obviously: $\sum_i \gamma_i = \sum_i \alpha_i + \sum_i \beta_i$.
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- (These three conditions above are also sufficient when the matrices are 2 \( \times \) 2.)
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- K. Fan, 1949: $\forall i < n, \sum_{i \leq r} \gamma_i \leq \sum_{i \leq r} \alpha_i + \sum_{i \leq r} \beta_i$. 
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- V. B. Lidskii, 1950, and H. Wielandt, 1955: $\forall l \not\subset \{1, \ldots, n\}$, $\sum_{i \in l} \gamma_i \leq \sum_{i \in l} \alpha_i + \sum_{i \leq \|l\|} \beta_i$. 
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- A. Horn, 1962: All conditions above, plus $\gamma_2 + \gamma_3 \leq \alpha_1 + \alpha_3 + \beta_1 + \beta_3$, are also sufficient for $3 \times 3$ matrices.
Observe that all necessary conditions above have the form

\[ \sum_{k \in K} \gamma_k \leq \sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j \]  

(*\text{IJK})

for certain subsets \( I, J, K \) of \( \{1, \ldots, n\} \).
Observe that all necessary conditions above have the form

\[ \sum_{k \in K} \gamma_k \leq \sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j \quad (*)_{IJK} \]

for certain subsets \( I, J, K \) of \( \{1, \ldots, n\} \).

A. Horn, 1962: If we “collect” all conditions in the above form, do we get a necessary and sufficient condition?
Of course we have to carefully collect \((\ast_{IJK})\)'s. Horn conjectured that the collection \(\bigcup_{r<n} T_r\) works, where \(T_r\) is defined inductively:
Of course we have to carefully collect \(( \ast_{IJK} )\)'s. Horn conjectured that the collection \(\bigcup_{r < n} T_r\) works, where \(T_r\) is defined inductively:

\[
T_1 := \{(I, J, K) \mid \sum I + \sum J = \sum K + 1\}
\]

\[
T_r := \left\{(I, J, K) \mid \begin{array}{l}
\sum I + \sum J = \sum K + \frac{r(r+1)}{2}; \\
\forall p < r, \forall (F, G, H) \in T_p, \\
\sum_{f \in F} i_f + \sum_{g \in G} j_g \leq \sum_{h \in H} k_h + \frac{p(p+1)}{2}
\end{array}\right\}
\]
Horn’s Conjecture

Horn’s Conjecture (proven by A. Knutson and T. Tao)

A triple \((\alpha, \beta, \gamma)\) occurs as eigenvalues of Hermitian matrices \(A, B\) and \(C = A + B\) iff. the following two conditions hold:

1. \(\sum \gamma_i = \sum \alpha_i + \sum \beta_i\), and
2. the inequalities

\[
\sum_{k \in K} \gamma_k \leq \sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j \quad (\ast_{IJK})
\]

holds for all \((I, J, K)\) in \(T_r\) for all \(r < n\).
Connections with LR Coefficients

At the level of combinatorics, the connection is:

Theorem (A. Knutson, T. Tao:)

If \( \alpha, \beta, \gamma \) are partitions, then \( c_{\gamma}^{\alpha,\beta} > 0 \) iff.

- \( \sum_i \gamma_i = \sum_i \alpha_i + \sum_i \beta_i, \) and
- the inequalities

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\sum_{k \in K} \gamma_k \leq \sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j \quad (\ast_{IJK})
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**Theorem (A. Knutson, T. Tao:)**

If $\alpha, \beta, \gamma$ are partitions, then $c^\gamma_{\alpha,\beta} > 0$ iff.

- $\sum_i \gamma_i = \sum_i \alpha_i + \sum_i \beta_i$, and
- the inequalities

\[ \sum_{k \in K} \gamma_k \leq \sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j \tag{\ast IJK} \]

holds for all $(I, J, K)$ in $T_r$ for all $r < n$.

The problem of Hermitian matrices can be related to Schubert calculus. I’ll try to explain this later.
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An Exercise from Hatcher

Exercise 2.1.14 from A. Hatcher’s *Algebraic Topology*:
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Exercise

Determine whether there exists a short exact sequence

\[ 0 \rightarrow \mathbb{Z}/4 \rightarrow \mathbb{Z}/8 \oplus \mathbb{Z}/2 \rightarrow \mathbb{Z}/4 \rightarrow 0. \]
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**Exercise**

Determine whether there exists a short exact sequence

\[ 0 \rightarrow \mathbb{Z}/4 \rightarrow \mathbb{Z}/8 \oplus \mathbb{Z}/2 \rightarrow \mathbb{Z}/4 \rightarrow 0. \]

Let \( \mathbb{Z}/4 \hookrightarrow \mathbb{Z}/8 \oplus \mathbb{Z}/2 \) be defined by \( 1 \mapsto (2, 1) \).
An Exercise from Hatcher

Exercise (continued)

More generally, determine which abelian groups $A$ fit into a short exact sequence

$$0 \longrightarrow \mathbb{Z}/p^m \longrightarrow A \longrightarrow \mathbb{Z}/p^n \longrightarrow 0.$$
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Some homological algebra $\implies A = \mathbb{Z}/p^{m+n}$ or $\mathbb{Z}/p^a \oplus \mathbb{Z}/p^{m+n-b}$, where $a \leq \min\{m, n\}$. 
An Exercise from Hatcher

Exercise (continued)

More generally, determine which abelian groups $A$ fit into a short exact sequence

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Some homological algebra $\implies A = \mathbb{Z}/p^{m+n}$ or $\mathbb{Z}/p^a \oplus \mathbb{Z}/p^{m+n-b}$, where $a \leq \min\{m, n\}$. $A$ is called an extension of $\mathbb{Z}/p^n$ by $\mathbb{Z}/p^m$. 
Extension Problem

A more general question is:

Question

Given finite abelian $p$-groups $\bigoplus_i \mathbb{Z}/p^{\alpha_i}$, $\bigoplus_i \mathbb{Z}/p^{\beta_i}$, which abelian group $A$ can be the extension of them? In other words, for which $A$ can we have a short exact sequence

$$
0 \longrightarrow \bigoplus_i \mathbb{Z}/p^{\alpha_i} \longrightarrow A \longrightarrow \bigoplus_i \mathbb{Z}/p^{\beta_i} \longrightarrow 0 ?
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Extension Problem

A more general question is:

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Given finite abelian $p$-groups $\bigoplus_i \mathbb{Z}/p^{\alpha_i}$, $\bigoplus_i \mathbb{Z}/p^{\beta_i}$, which abelian group $A$ can be the extension of them? In other words, for which $A$ can we have a short exact sequence

$$0 \rightarrow \bigoplus_{i} \mathbb{Z}/p^{\alpha_i} \rightarrow A \rightarrow \bigoplus_{i} \mathbb{Z}/p^{\beta_i} \rightarrow 0 ?$$

(One can easily compute $\text{Ext}^1_{\mathbb{Z}}(\bigoplus_i \mathbb{Z}/p^{\alpha_i}, \bigoplus_i \mathbb{Z}/p^{\beta_i})$, which classifies all abelian group extensions. But this doesn’t tell us what those extensions are.)
Observations

Let’s make some first step observations.

\[ 0 \longrightarrow \bigoplus_{i} \mathbb{Z}/p^{\alpha_i} \longrightarrow A \longrightarrow \bigoplus_{i} \mathbb{Z}/p^{\beta_i} \longrightarrow 0 \]
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\[
0 \rightarrow \bigoplus_i \mathbb{Z}/p^{\alpha_i} \rightarrow A \rightarrow \bigoplus_i \mathbb{Z}/p^{\beta_i} \rightarrow 0
\]

- By counting elements, we know \( A \) is necessarily a \( p \)-group. So \( A = \bigoplus_i \mathbb{Z}/p^{\gamma_i} \) for some \( \gamma \).
Observations

Let's make some first step observations.

\[0 \rightarrow \bigoplus_{i} \mathbb{Z}/p^{\alpha_i} \rightarrow A \rightarrow \bigoplus_{i} \mathbb{Z}/p^{\beta_i} \rightarrow 0\]

- By counting elements, we know \( A \) is necessarily a \( p \)-group. So \( A = \bigoplus_{i} \mathbb{Z}/p^{\gamma_i} \) for some \( \gamma \).
- Any finite abelian \( p \)-group determines, and is determined by a partition:

\[M_{\gamma} := \bigoplus_{i} \mathbb{Z}/p^{\gamma_i} \iff \gamma = (\gamma_1 \geq \gamma_2 \geq \cdots)\]

\( \gamma \) is called the type of \( M_{\gamma} \).
Observation 1

Let $G_{\alpha,\beta}^\gamma(p)$ denote the number of non-isomorphic extensions.

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- The trivial observation: $M_{\alpha} \subseteq M_{\gamma}$. 
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Let $G_{\alpha,\beta}^\gamma(p)$ denote the number of non-isomorphic extensions.

$$0 \rightarrow M_\alpha \rightarrow M_\gamma \rightarrow M_\beta \rightarrow 0$$

- The trivial observation: $M_\alpha \subseteq M_\gamma$.
  
  $G_{\alpha,\beta}^\gamma(p) > 0 \implies \alpha \subseteq \gamma.$
Observation 2

\[ 0 \rightarrow M_\alpha \rightarrow M_\gamma \rightarrow M_\beta \rightarrow 0 \]
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\[ 0 \rightarrow M_\alpha \rightarrow M_\gamma \rightarrow M_\beta \rightarrow 0 \]

- The length of these groups must add up:

\[ \ell(M_\gamma) = \ell(M_\alpha) + \ell(M_\beta). \]
Observation 2

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The length of these groups must add up:
\[
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\]

One way to compute length:
\[
\ell(M) = \sum_{i \geq 1} \dim_{\mathbb{Z}/p}(p^{i-1}M/p^i M).
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In fact, \( \dim_{\mathbb{Z}/p}(p^{i-1}M_\alpha/p^i M_\alpha) = \alpha_i^T. \)
Observation 2

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One way to compute length:

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In fact, \( \dim_{\mathbb{Z}/p}(p^{i-1}M_\alpha/p^i M_\alpha) = \alpha_i^T. \)

\[ G_{\alpha, \beta}^\gamma(p) > 0 \quad \Rightarrow \quad |\gamma^T| = |\alpha^T| + |\beta^T| \quad \Rightarrow \quad |\gamma| = |\alpha| + |\beta|. \]
Observations

These conditions is the same as $\alpha \subseteq \gamma$ and $|\gamma \setminus \alpha| = |\beta|$, a necessary condition for $c_{\alpha,\beta}^\gamma$ to be nonzero.
Relation to LR Coefficients

By manipulating lengths (which involves long combinatorial calculation) and some group theory (e.g. Pontryagin duality), one can show:

**Theorem**

> Given an extension $0 \to M_\alpha \to M_\gamma \to M_\beta \to 0$, for each $i \geq 0$, let $\gamma^{(i)}$ be the type of $M_\gamma / p^i M_\alpha$. Then the sequence

$$\gamma^{(0)} T \subseteq \gamma^{(1)} T \subseteq \ldots \subseteq \gamma^{(r)} T$$

(where $p^r M_\alpha = 0$) corresponds to an LR tableau for the triple $(\alpha, \beta, \gamma)$. 
Relation to LR Coefficients
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- Each difference $\gamma^{(i)}T \setminus \gamma^{(i-1)}T$ has only one box at each column.
  $\Rightarrow$ filling in $i$ in $\gamma^{(i)}T \setminus \gamma^{(i-1)}T$ gives us a tableau whose rows are weakly increasing, columns are strongly increasing.
As a result

**Corollary**

\[(c^{\gamma T}_{\alpha T, \beta T}) c^{\gamma}_{\alpha, \beta} = 0 \implies G^{\gamma}_{\alpha, \beta} = 0.\]

In fact, more is true:

**Theorem**

*For any triple* \(\alpha, \beta, \gamma\), \(\exists! g^{\gamma}_{\alpha, \beta} \in \mathbb{Z}[t]\), *such that* \(G^{\gamma}_{\alpha, \beta}(p) = g^{\gamma}_{\alpha, \beta}(p)\).*

*In fact, if* \(c^{\gamma}_{\alpha, \beta} = 0\) *then* \(g^{\gamma}_{\alpha, \beta} = 0\); *otherwise,*

\[g^{\gamma}_{\alpha, \beta}(t) = c^{\gamma}_{\alpha, \beta}(t - 1)^N + a_{N-1}(t - 1)^{N-1} + \cdots + a_0\]

*for some* \(a_i \in \mathbb{Z}_{\geq 0}\).*
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Therefore \(c^\gamma_{\alpha, \beta} > 0 \iff G^\gamma_{\alpha, \beta}(p) > 0\) *for some* \(p\)

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Generalizations

This theorem is still true if we instead consider finite length modules over a discrete valuation ring.
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1. Defining LR Coefficients
2. Decomposing Representations of $GL_n$
3. Eigenvalues of Hermitian Matrices
4. Extensions of Finite Abelian $p$-Groups
5. Intersection of Schubert Cells
6. Underlying Connections
Grassmannians

**Definition (Grassmannian)**

Let $V$ be an $n$-dimensional vector space (say over $\mathbb{C}$), $0 \leq r \leq n$. The Grassmannian $\text{Gr}_r(V)$ is the set of $r$-dimensional subspaces of $V$. 

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There is a way to give this an analytic structure. With this structure $\text{Gr}_r(V)$ is a compact complex manifold of dimension $r(n - r)$. 
Schubert Decomposition

There are some nice subspaces in $\text{Gr}_r(V)$. Fix a basis $E = \{e_1, \ldots, e_n\}$ of $V$. For any strictly increasing sequence $P = (1 \leq p_1 < p_2 < \cdots < p_r \leq n)$, define the Schubert cell corresponding to $P$ to be

$$X(E)_P := \{L \in \text{Gr}_r(X) \mid \forall i, \dim(L \cap \text{span}\{e_1, \ldots, e_i\}) \geq i\}.$$  

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In general these subspaces have singularities. As before, we want to parametrize these objects using partitions. Letting $\lambda_j := n - r + j - p_j$, $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r)$ is a partition. Let $X(E)_\lambda := X(E)_P$. 
Schubert Decomposition

These subspaces are nice because of the following. \( \text{GL}(V) \subset V \), so \( B = \{ \text{upper-} \Delta \text{ matrices} \} \subset \text{GL}(V) \subset \text{Gr}_r(V) \), and \( \text{Gr}_r(V) \) is a disjoint union of orbits under the action of \( B \).

**Theorem**

Schubert Decomposition For each \( \lambda \), there is an orbit \( X(E)_\lambda^\circ \) isomorphic to \( \mathbb{C}^{r(n-r)-|\lambda|} \) such that \( \overline{X(E)_\lambda^\circ} = X(E)_\lambda \).

Also, \( \text{Gr}_r(V) = \bigsqcup_{\lambda \subseteq r \times (n-r)} X_\lambda^\circ \).
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In topological point of view, this gives us a CW structure on $Gr_r(V)$, and $X(E)_{\lambda}$ is a $2(r(n-r) - |\lambda|)$-cell. One can compute cohomologies using this decomposition.
Intersections

People want to understand the intersection between the cells. A basic result is:

**Proposition**

\[ X(E)_\lambda = \bigsqcup_{\mu \subseteq \lambda} X(E)^\circ_\mu = \bigcup_{\mu \subseteq \lambda} X(E)_\mu. \]
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What we want to understand is when the two cells \( X(E)_{\lambda} \) and \( X(F)_{\mu} \) are defined using different bases \( E \neq F \).
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**Question**

Given two distinct bases \( E, F \) of \( V \) and two partitions \( \lambda, \mu \subseteq r \times (n - r) \) of length \( r \), what’s the intersection \( X(E)_\lambda \cap X(F)_{\mu} \)?
Cohomology

To study this question, we need to translate it to something we can work on.
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Cohomology

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Things are easier to compute here, because $\bigoplus_i H^i(\text{Gr}_r(V))$ is not just a group, but also a ring (under the cup product).
Cohomology

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Things are easier to compute here, because $\bigoplus_i H^i(\text{Gr}_r(V))$ is not just a group, but also a ring (under the cup product).

**Theorem**

*If $E$ and $F$ are different bases of $V$, then*

$$[X_\lambda] \sim [X_\mu] = [X(E)_\lambda \cap X(F)_\mu].$$
Connection with LR Coefficients

Theorem

\[ [X_\alpha] \sim [X_\beta] = \sum_{\gamma \subseteq r \times (n-r)} c_{\alpha,\beta}^{\gamma} [X_\gamma]. \]
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\[ [X_\alpha] \sim [X_\beta] = \sum_{\gamma \subseteq r \times (n-r)} c_{\alpha,\beta}^\gamma [X_\gamma]. \]

What this means is that, for generic choice of distinct bases \( E, F, G \) of \( V \), the intersection

\[ X(E)_\alpha \cap X(F)_\beta \cap X(G)_\gamma \]

consists of \( c_{\alpha,\beta}^\gamma \) many points.
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Connections

Representations of $\text{GL}(V)$

Intersection of Schubert Cells

Extensions of finite abelian $p$-groups

Eigenvalues of Hermitian matrices
Connections

- Representations of $GL(V)$
- Intersection of Schubert Cells
- Extensions of finite abelian $p$-groups
- Eigenvalues of Hermitian matrices
A connection between these two areas is via the proof of the theorem

**Theorem**

\[
[X_\alpha] \sim [X_\beta] = \sum_{\gamma \subseteq r \times (n-r)} c_{\alpha,\beta}^\gamma [X_\gamma].
\]

It goes as follows:
Proof of the theorem

For each $S_\lambda V$, we can get a polynomial:

$$s_\lambda = \text{Tr} \rho_\lambda \text{diag}(x_1, \ldots, x_n)$$

(where any $A \in \text{GL}(V)$ acts on $S_\lambda V$ by the matrix $\rho_\lambda A$), called the \textbf{character} of $S_\lambda V$. 
Proof of the theorem

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$$S_{\lambda} V \otimes S_{\mu} V = \bigoplus_{\nu} (S_{\nu} V)^{\oplus c_{\lambda,\mu}^\nu} \iff s_{\lambda} \cdot s_{\mu} = \sum_{\nu} c_{\lambda,\mu}^\nu s_{\nu}.$$
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- $S_\lambda V \otimes S_\mu V = \bigoplus_{\nu} (S_\nu V)^{\oplus c_\nu^{\lambda,\mu}} \leadsto s_\lambda \cdot s_\mu = \sum_{\nu} c_\nu^{\lambda,\mu} s_\nu$.

- $\{s_\lambda\}_\lambda$ form a basis of the ring $\Lambda$ of all symmetric polynomials with integer coefficients (i.e. $\Lambda = \mathbb{C}[x_1, \ldots, x_n]^{S_n}$).
Proof of the theorem

- Show that \([X_\lambda]\)\_\lambda satisfies the same product rule as \(s_\lambda\)\_\lambda does.
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- Show that \( \{[X_\lambda]\}_\lambda \) satisfies the same product rule as \( \{s_\lambda\}_\lambda \) does.

- Show that \( \Lambda \to H^*(\text{Gr}_r(V)) \), \( s_\lambda \mapsto [X_\lambda] \) is a ring isomorphism.
Proof of the theorem

- Show that $\{[X_\lambda]\}_\lambda$ satisfies the same product rule as $\{s_\lambda\}_\lambda$ does.
- Show that $\Lambda \rightarrow H^\ast(\text{Gr}_r(V))$, $s_\lambda \mapsto [X_\lambda]$ is a ring isomorphism.

PolyRep($\text{GL}(V)$) $\longleftrightarrow \Lambda \longleftrightarrow H^\ast(\text{Gr}_r(V))$. 
Reps of $\mathbf{GL}(V)$ $\leftrightarrow$ Schubert Cells

One would expect a connection without going through the combinatorics, since we have a natural action $B \subseteq \mathbf{GL}(V) \acts \mathbf{Gr}_r(V)$. 
Reps of $\mathbf{GL}(V)$ $\leftrightarrow$ Schubert Cells

One would expect a connection without going through the combinatorics, since we have a natural action $B \subset \mathbf{GL}(V) \trianglerighteq \mathbf{Gr}_r(V)$. But such a connection (that’s at the same time conceptually satisfying) will use some higher machineries. There isn’t a easy way to describe this.
Connections

- Representations of $\text{GL}(V)$
- Intersection of Schubert Cells
- $C_{\alpha,\beta}^\gamma$
- Extensions of finite abelian $p$-groups
- Eigenvalues of Hermitian matrices
Schubert Cells ⟷ Hermitian Matrices

The key to relating these two areas is **Rayleigh trace**. If $A$ is an $n \times n$ Hermitian matrix, then for any subspace $L \subseteq \mathbb{C}^n$, the Rayleigh trace is defined to be

$$R_A(L) := \text{Tr}(L \hookrightarrow \mathbb{C}^n \xrightarrow{A} \mathbb{C}^n \xrightarrow{\text{pr}} L).$$
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$$R_A(L) := \text{Tr}(L \hookrightarrow \mathbb{C}^n \xrightarrow{A} \mathbb{C}^n \xrightarrow{\text{pr}} L).$$

Let $\alpha = (\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n)$ be eigenvalues of $A$ and $v_1, \ldots, v_n$ the corresponding eigenvectors. The $v_i$’s form an ordered basis $E(A)$ of $\mathbb{C}^n$. 

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**Schubert Cells ↔ Hermitian Matrices**
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Theorem (J. Hersch and B. Zwahlen)

For any subset \( P = \{p_1 < \cdots < p_r\} \subseteq \{1, \ldots, n\} \),

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\sum_{i \in I} \alpha_i = \min_{L \in X(E(A))_P} R_A(L).
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Schubert Cells $\leftrightarrow$ Hermitian Matrices

**Theorem (J. Hersch and B. Zwahlen)**

For any subset $P = \{p_1 < \cdots < p_r\} \subseteq \{1, \ldots, n\}$,

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Using this result one can go back and forth between the two areas.
Connections

- Representations of $GL(V)$
- Intersection of Schubert Cells
- $c_{\alpha,\beta}^{\gamma}$
- Extensions of finite abelian $p$-groups
- Eigenvalues of Hermitian matrices
The LR coefficients measure some kind of “multiplicity” in Rep theory, Schubert calculus and $p$-group extensions problems. Is there a similar interpretation in Hermitian eigenvalue problems?
The LR coefficients measure some kind of “multiplicity” in Rep theory, Schubert calculus and $p$-group extensions problems. Is there a similar interpretation in Hermitian eigenvalue problems? Knutson: $c_{\alpha,\beta}^{\gamma}$ is the asymptotic “volume” of $\{(A, B, C = A + B) \mid A, B, C \text{ have respective eigenvalues } N_{\alpha}, N_{\beta}, N_{\gamma}\}$ as $N \to \infty$. 
The End