A NON-INTEGRAL KAZHDAN-LUSZTIG ALGORITHM AND APPLICATION TO WHITTAKER MODULES

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Abstract. Let $\mathfrak{g}$ be a complex semisimple finite dimensional Lie algebra, and consider a category of representations of $\mathfrak{g}$ where a Kazhdan-Lusztig algorithm exists for integral regular infinitesimal characters. In this talk, we will discuss a potential approach for extending the integral algorithm to arbitrary non-integral regular infinitesimal characters, using intertwining functors. We will then apply this approach to Whittaker modules and demonstrate the non-integral algorithm there using an explicit example.

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Thank you very much for the introduction and for inviting me to speak here.

So what I want to talk about is a Kazhdan-Lusztig algorithm for Whittaker modules for non-integral infinitesimal characters, which is an extension of the integral algorithm proven by Anna Romanov. Historically there are a few ways of extending integral KL type results to the non-integral case. The method I'm about to explain is different from the well-known ones, and it uses intertwining functors due to Beilinson-Bernstein. The idea of this method was already present in the literature, for example in David Vogan's papers on KL algorithm for real groups, but I think it deserves more attention and more use. So I'll spend the first half of the talk describing how this method works, phrasing everything in a slightly general language, and then in the second half, we will apply this method to Whittaker modules.

1. General scheme

Notation 1.1. Let

- $\mathfrak{g}$ = complex semisimple Lie algebra; G = corresponding group.
- $\mathfrak{h}$ = universal Cartan algebra;
- $\lambda \in \mathfrak{h}^*$ is regular, giving rise to a regular infinitesimal character.
- X = flag variety of $\mathfrak{g}$.
- $\mathcal{D}_\lambda$ = tdo on X with infinitesimal character λ.
- C_λ = the category of reps we are doing KL algorithm in. For simplicity we will consider $\text{Mod}_{fg}(\mathcal{U}(\mathfrak{g})_\lambda, N)$ (highest weight modules) and $\text{Mod}_{fg}(\mathcal{U}(\mathfrak{g})_\lambda, N, \eta)$ (Whittaker modules).
- $\mathcal{C}_\lambda$ = localization of C, i.e. $\text{Mod}_{coh}(\mathcal{D}_\lambda, N)$ or $\text{Mod}_{coh}(\mathcal{D}_\lambda, N, \eta)$.

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- $\mathcal{L}(Q)$ = irreducible objects in $\mathcal{C}$ (Q denotes an N-orbit in X, and τ is a K-equivariant connection on Q compatible with the infinitesimal character). In the cases we are considering, there is only one connection on each orbit, so we may abbreviate $\mathcal{L}(Q)$.
- $\mathcal{I}(Q) = \mathcal{I}(Q) = i_{Q*}\mathcal{O}_Q$ = standard objects, containing $\mathcal{L}(Q)$ as its unique irreducible sub.
- W = Weyl group of $\mathfrak{g}$;
- $\mathcal{H}$ = Hecke algebra of W. Let $\{T_w\}$ be the standard basis and $\{C_w\}$ be the Kazhdan-Lusztig basis.

Goal of KL Algorithm 1.2. *is to write $[\mathcal{L}(Q)]$ as a linear combination of the $[\mathcal{I}(Q)]$'s in the Grothendieck group $K\mathcal{C}$.*

1.1. **Integral case.** Before we discuss the non-integral case, let's first recall the argument for integral infinitesimal characters and setup the notations. A lot of you are probably already familiar with this, so please bear with me. There are a few different formulations of the algorithm. I will follow the version that I'm most comfortable with.

It's easy to see that $K\mathcal{C}$ is actually a free abelian group, and the $[\mathcal{I}(Q)]$'s form a basis:

$$\begin{array}{ccc} \mathcal{C} & & \\ \downarrow{[-]} & & \\ K\mathcal{C} & \longrightarrow & \mathbb{Z}\cdot\{T(Q)\}_{Q\in N\backslash X} \\ [\mathcal{I}(Q)] & \longmapsto & T(Q) \end{array}$$

This is not enough to find the images of irreducibles. The first step is to construct a $\mathbb{Z}[q^{\pm 1}]$-module that enriches the Grothendieck group.

$\mathbb{Z}[q^{\pm 1}]$-module. The enriched version is a free $\mathbb{Z}[q^{\pm 1}]$-module over the same basis. We also want to lift the bottom map to a comparison map which we denote by ν, defined as follows. Let $i_Q : Q \hookrightarrow X$ be the inclusion map of an N-orbit. Then for any $\mathcal{V} \in \mathcal{C}$, the $\mathcal{D}$-module theoretic pullback $i_Q^!$ is a semisimple complex (is a direct sum of connections at various degrees), so we may take its generating function in variable q, denoted by $\chi_q(i_Q^!\mathcal{V})$. Then the comparison map may be defined as

$$\begin{aligned} \mathcal{C} &\xrightarrow{\nu} \mathcal{E}, \\ \mathcal{V} &\mapsto \sum_Q \chi_q(i_Q^!\mathcal{V})T(Q)\ , \\ \mathcal{I}(Q) &\mapsto T(Q), \\ \mathcal{L}(Q) &\mapsto C(Q). \end{aligned}$$

At $q = -1$, each $\chi_q(i_{Q'}^!\mathcal{L}(Q))$ is additive on short exact sequences, and hence ν factors through the Grothendieck group, as desired. $\chi_q(i_{Q'}^!\mathcal{L}(Q))$ is then the coefficient of $T(Q')$ in $C(Q)$, i.e. Kazhdan-Lusztig polynomials.

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\nu} & \mathcal{E} \\ \downarrow{[-]} & & \downarrow{q=-1} \\ K\mathcal{C} & \xrightarrow{\cong} & \mathbb{Z}\cdot\{T(Q)\} \end{array}.$$

Geometric Hecke action. In order to obtain an algorithm, one also need an $\mathcal{H}$-action on $\mathcal{E}$ and lift this action to a "geometric action" on $\mathcal{C}$ on standard and irreducible objects. In the highest weight case, $\mathcal{E} = \mathcal{H}$ and the action $\mathcal{H} \circlearrowleft \mathcal{H}$ is just the (right) regular action. In general, the action $\mathcal{E} \circlearrowleft \mathcal{H}$ is modeled based on the geometric action of $\mathcal{H}$ on standard modules: For each $w \in W$, let Z_w be the diagonal G-orbit in $X \times X$ corresponding to pairs of Borels in relative position w, with projection maps

$$X \xleftarrow{p_1} Z_w \xrightarrow{p_2} X.$$

Definition 1.3 (Beilinson-Bernstein)**.** The **intertwining functor** LI_w (due to Beilinson-Bernstein) is defined by pulling-pushing along the projection maps:

$$LI_w : D^b(\mathcal{D}_\lambda) \to D^b(\mathcal{D}_{w\lambda}),$$
$$\mathcal{V}^\bullet \mapsto p_{1+}\Big(p_1^*\mathcal{O}_X(\rho - w\rho) \underset{\mathcal{O}_{Z_w}}{\otimes} p_2^+\mathcal{V}^\bullet\Big).$$

Here the twist $p_1^*\mathcal{O}_X(\rho - w\rho)$ is to ensure the functor lands into the correct category. As the notation suggests, LI_w is the left derived functor of $I_w := H^0 LI_w$. In fact, LI_w is an equivalence of (derived) categories.

One than defines $\mathcal{E} \circlearrowleft \mathcal{H}$ by

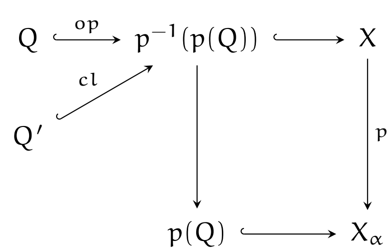
$$T(Q) \cdot T_w := \nu\big(\mathcal{O}_X(\lambda - w\lambda) \otimes LI_w \mathcal{I}(Q)\big).$$

In the Grothendieck group (at $q = -1$) this is the *coherent continuation* representation of $\mathcal{H}|_{q=-1} = \mathbb{C}[W]$. One can verify that the the action of the KL basis elements $C_{s_\alpha} \in \mathcal{H}$ can also be lifted geometrically:

$$T(Q) \cdot C_{s_\alpha} = \nu\big(U_\alpha \mathcal{I}(Q)\big)$$

where U_α is defined by pulling-pushing along the natural map $X \to X_{s_\alpha}$ to the partial flag variety of type s_α.

Algorithm. With these actions, one can find $C(Q)$ as follows by induction on $\dim Q$. $\mathcal{L}(Q)$ is either equal to $\mathcal{I}(Q)$ (in which case $C(Q) = T(Q)$ is known) or reducible. In the latter case, there is a simple root α and an N-orbit $Q' \subset \overline{Q}$ so that Q' is closed in $p_\alpha^{-1}(p_\alpha(Q'))$ and Q is open in it, and the connection τ' on Q' is so that $\mathcal{L}(Q)$ appears in $U_\alpha \mathcal{L}(Q')$, with $\dim Q' = \dim Q - 1$ (we say that α is **transversal** to Q' and **non-transversal** to Q).



One shows (possibly nontrivially, also by induction) that

$$C(Q') \cdot C_{s_\alpha} = \nu(\mathcal{L}(Q')) \cdot C_{s_\alpha} = \nu(U_\alpha \mathcal{L}(Q')).$$

Since by induction assumption $C(Q')$ is known, the left side can be computed explicitly in $\mathcal{E}$. For the right side, by the (deep) Decomposition Theorem, $U_\alpha \mathcal{L}(Q')$ is a direct sum of irreducible objects with $\dim \mathrm{Supp} \leqslant \dim Q$, and $\mathcal{L}(Q)$ is the only one with $\mathrm{Supp} = \overline{Q}$. Therefore the equation above together with known information on the $C(Q')$'s allows us to solve for $C(Q)$.

For other categories of representations (for example $(\mathfrak{g}, K)$-modules), this last step may require more work, but the process should be the similar.

1.2. **Non-integral case.** Now assume λ is regular but not necessarily integral. Let

- $W_\lambda =$ the integral Weyl group (as a subgroup of W);
- $\mathcal{H}_\lambda =$ Hecke algebra of W_λ.
- To emphasize the infinitesimal character we write $\mathcal{C}_\lambda = \mathcal{C}$ and $\mathcal{E}_\lambda = \mathcal{E}$.
- Standard and irreducible objects will be written as $\mathcal{I}(Q, \lambda)$ and $\mathcal{L}(Q, \lambda)$.

In this case the action $\mathcal{E} \circlearrowleft \mathcal{H}$ should be replaced by $\mathcal{E}_\lambda \circlearrowleft \mathcal{H}_\lambda$, which can be defined in the same way as before since the intertwining functors are still defined. In order to run the same argument, one would like to realize right multiplication of C_{s_α} geometrically for any simple reflection s_α in W_λ. However, s_α in W_λ may be non-simple in W, and in this case we can't define U_α as before (because there is no X_{s_α} anymore).

Non-integral reflection. To remedy this, observe that

Observation 1.4. *Suppose s_α is simple in W_λ and*

$$s_\alpha = s_{\beta_1} \cdots s_{\beta_k} s_\gamma s_{\beta_k} \cdots s_{\beta_1}$$

is a reduced expression in W, then

- *Each s_{β_i} is non-integral to $s_{\beta_{i-1}} \cdots s_{\beta_1}\lambda$, and*
- *s_γ is (simple and) integral to $s_{\beta_k} \cdots s_{\beta_1}\lambda$.*

The idea then is to "apply" the s_{β_i}'s, translate everything from λ to $s_{\beta_k} \cdots s_{\beta_1}\lambda$, apply U_α, and translate back. In order to do this geometrically, we need

Proposition 1.5 (Beilinson-Bernstein). *If s_β is simple and is non-integral to λ, then the 0-th intertwining functor*

$$I_{s_\beta} = H^0 LI_{s_\beta} : \mathrm{Mod}(\mathcal{D}_\lambda) \to \mathrm{Mod}(\mathcal{D}_{s_\beta\lambda})$$

is an equivalence of categories. Its inverse is I_{s_β}.

We will call I_{s_β} a **non-integral intertwining functor**. Moreover, it is not hard to show that I_{s_β} sends standard modules to standard modules. We denote the image by

$$I_{s_\beta} : \mathcal{I}(Q, \lambda) \mapsto \mathcal{I}(Qs_\beta, s_\beta\lambda).$$

Since I_{s_β} is an equivalence of categories, it sends the unique irreducible sub $\mathcal{L}(Q, \lambda)$ of $\mathcal{I}(Q, \lambda)$ to the unique irreducible sub $\mathcal{L}(Qs_\beta, s_\beta\lambda)$ of $\mathcal{I}(Qs_\beta, s_\beta\lambda)$:

$$I_{s_\beta} : \mathcal{L}(Q, \lambda) \mapsto \mathcal{L}(Qs_\beta, s_\beta\lambda).$$

Moreover, the Kazhdan-Lusztig polynomials are also preserved under I_{s_β}, namely

$$\chi_q i^!_{Q' s_\beta} \mathcal{L}(Qs_\beta, s_\beta\lambda) = \chi_q i^!_{Q'} \mathcal{L}(Q, \lambda).$$

We may summarize these in the following diagram

$$\begin{array}{ccccccc}
\mathcal{C}_\lambda & \xrightarrow{\nu} & \mathcal{E}_\lambda & = & \bigoplus_j \mathcal{E}_{\lambda,j} & T(Q,\lambda) & C(Q,\lambda) \\
\cong \downarrow I_{s_\beta} & & \downarrow & & \downarrow & \downarrow & \downarrow \\
\mathcal{C}_{s_\beta\lambda} & \xrightarrow{\nu} & \mathcal{E}_{s_\beta\lambda} & = & \bigoplus_j \mathcal{E}_{s_\beta\lambda,j} & T(Qs_\beta, s_\beta\lambda) & C(Qs_\beta, s_\beta\lambda)
\end{array}$$

where the vertical maps on the middle and on the right intertwines the actions of

$$\mathcal{H}_\lambda \xrightarrow{\sim} \mathcal{H}_{s_\beta\lambda}, \quad T_w \mapsto T_{s_\beta w s_\beta}.$$

The arrow on the left also intertwines the geometric lifts of the Hecke actions.

Geometric Hecke action. For a reflection s_α simple in W_λ and a reduced expression $s_\alpha = s_{\beta_1} \cdots s_{\beta_k} s_\gamma s_{\beta_k} \cdots s_{\beta_1}$ in W, the action of T_{s_α} on $T(Q,\lambda)$ before can be rewritten as

$$\begin{aligned} T(Q,\lambda) \cdot T_{s_\alpha} &:= \nu\big(\mathcal{O}_X(\lambda - s_\alpha\lambda) \otimes LI_{s_\alpha}\mathcal{I}(Q,\tau)\big) \\ &= \nu\big(\mathcal{O}_X(\lambda - s_\alpha\lambda) \otimes I_{s_{\beta_1}} \cdots I_{s_{\beta_k}} LI_{s_\gamma} I_{s_{\beta_k}} \cdots I_{s_{\beta_1}} \mathcal{I}(Q,\tau)\big). \end{aligned}$$

Multiplication by $C_{s_\alpha} \in \mathcal{H}_\lambda$ can now be lifted to

$$T(Q,\tau) \cdot C_{s_\alpha} = \nu\big(I_{s_{\beta_1}} \cdots I_{s_{\beta_k}} U_\gamma I_{s_{\beta_k}} \cdots I_{s_{\beta_1}} \mathcal{I}(Q,\tau)\big).$$

Algorithm. At this point it's clear how the algorithm should be modified to the non-integral situation. Suppose by induction we know how to compute $C(Q',\lambda')$ for all $\dim Q' < \dim Q$ and all $\lambda' \in W \cdot \lambda$. Consider $C(Q,\lambda)$. If it is not equal to a standard module, then there is a simple reflection s not transversal to Q. If $s = s_\alpha$ is integral to λ, then one proceeds as in the integral case with U_α. If $s = s_\beta$ is non-integral to λ, then one applies I_{s_β} and $\mathcal{L}(Q,\lambda) = I_{s_\beta}\mathcal{L}(Qs_\beta, s_\beta\lambda)$ where $\dim(Qs_\beta) < \dim Q = k$. Since the Kazhdan-Lusztig polynomials of $\mathcal{L}(Qs_\beta, s_\beta\lambda)$ are known by induction, we know those of $\mathcal{L}(Q,\lambda)$ because I_{s_β} preserves KL polynomials as well.

1.3. **Features of this method.**

(1) This is an entirely $\mathcal{D}$-module theoretic argument that does not seem to have parallels in the perverse sheaf language. Because of this, this method can be applied to categories of representations where the corresponding $\mathcal{D}$-modules are holonomic but not regular holonomic (for example, Whittaker modules).
(2) Because the non-integral intertwining functors are equivalences of categories between all quasi-coherent $\mathcal{D}$-modules (not just equivariant/holonomic ones), they have the potential to be applied to other categories of representations of $\mathfrak{g}$ for extending integral results to non-integral infinitesimal characters.

However, this method does not give us a block decomposition of the category at non-integral infinitesimal character, nor does it give you a character formula for the irreducibles in each block. What it *does* give you is an algorithm allowing you to compute examples. Once the pattern is found an a conjecture of block decomposition and character formula is written down, this method can be run again to prove them.

2. Application to Whittaker modules

2.1. **Backgrounds and the integral case.** Now we apply this argument to Whittaker modules. Let's first recall the basic facts of this category.

Let $\mathfrak{n}$ be the Lie algebra of N, let $\eta : \mathfrak{n} \to \mathbb{C}$ be a representation, and let

$$C_\lambda = \mathcal{N}_{\lambda,\eta} = \{\text{f.g. } \mathfrak{g}\text{-modules with inf. char. } \lambda \text{ and } \mathfrak{n} \text{ acts by generalized character } \eta\}.$$

This is a highest weight category (Brown-Romanov). If $\eta = 0$, we get back the highest weight modules. In general, write

$$\Theta = \{\text{simple roots } \alpha \text{ so that } \eta \text{ is nonzero on the } \alpha\text{-root space in } \mathfrak{n}\},$$

then the standard objects and irreducible objects are parameterized by $W_\Theta \backslash W$. For a coset $D \in W_\Theta\backslash W$, the corresponding standard module is

$$M(D\lambda,\eta) = \mathcal{U}(\mathfrak{g}) \underset{\mathcal{U}(\mathfrak{p}_\Theta)}{\otimes} \left(\mathcal{U}(\mathfrak{l}_\Theta)_{D\lambda+(\rho\text{-shifts})} \underset{\mathcal{U}(\mathfrak{n}_\Theta)}{\otimes} \mathbb{C}_\eta\right) \quad \text{(McDowell)},$$

and its unique irreducible quotient is denoted by $L(D\lambda, \eta)$ (McDowell).

Regarding $[M(D\lambda, \eta) : L(E\lambda, \eta)]$,

- λ integral: Miličić-Soergel.
- λ non-integral: Backelin (using ideas of Soergel modules).

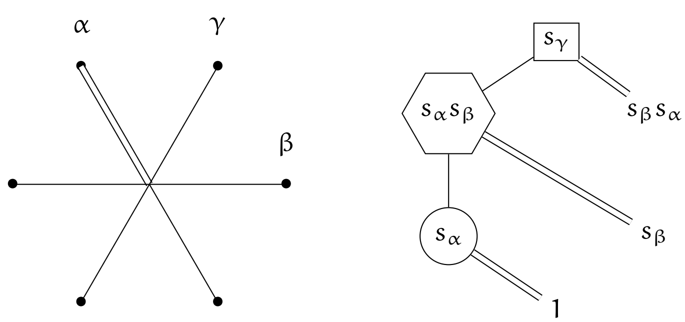
In order to get an algorithm for the KL polynomials themselves, we need to localize. The localization of $\mathcal{N}_{\lambda,\eta}$ is the category $\mathcal{C}_\lambda = \mathrm{Mod}_{coh}(\mathcal{D}_\lambda, N, \eta)$ consisting of weakly N-equivariant coherent $\mathcal{D}_\lambda$-modules so that the $\mathfrak{n} \subset \mathcal{D}_\lambda$-action differs from differential of N-action by η. These modules are

- holonomic but not regular holonomic;
- $M(D\lambda, \eta)$ is localized to $\mathcal{M}(D, \lambda, \eta) := i_{C(w^D)!}\mathcal{O}^\eta_{C(w^D)}$, where w^D is the longest element in D w.r.t. Bruhat order, $C(w^D)$ is the Schubert cell of w^D, and $\mathcal{O}^\eta_{C(w^D)}$ is the unique irreducible connection on $C(w^D)$ in this category. *There is no connection on Schubert cells not of the form* $C(w^D)$.
- Standard modules are denoted by $\mathcal{I}(D, \lambda, \eta)$ and $\mathcal{L}(D, \lambda, \eta)$, respectively. The duality $\mathbb{D}$ of holonomic $\mathcal{D}$-modules sends $\mathcal{M}$ to $\mathcal{I}$ and preserves $\mathcal{L}$. So it suffices to find KL polynomials between $\mathcal{L}$ and $\mathcal{I}$.
- The module $\mathcal{E} = \mathcal{H}_\Theta = \mathbb{Z}[q, q^{-1}] \cdot (W_\Theta \backslash W) \circlearrowleft \mathcal{H}$.

Using the approach outlined in the first half of the talk, Romanov showed

Theorem 2.1 (Int WKL, Romanov)**.** *If λ is integral regular, there is a KL algorithm for $\mathcal{N}_{\lambda,\eta}$. The WKL polynomials are the antispherical parabolic KL polynomials.*

Example 2.2 (A2)**.** $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$, $\Theta = \{\alpha\}$, $\lambda = -\rho$. There are three right W_Θ-cosets



	$T(s_\alpha)$	$T(s_\alpha s_\beta)$	$T(s_\gamma)$
$C(s_\alpha)$	1		
$C(s_\alpha s_\beta)$	q	1	
$C(s_\gamma)$		q	1

2.2. **Non-integral case.** Now consider non-integral λ. Because of the non-integrality of λ, our category is broken down into a direct sum of smaller blocks, and it turns out that the blocks are parameterized by doulbe cosets $W_\Theta \backslash W / W_\lambda$. The KL polynomials in each block are the same as antispherical PKL polynomials for an integral block.

To describe these integral blocks, we need some combinatorial facts of the double cosets.

Combinatorics of double cosets. The first fact we need is

- Each double coset $W_\Theta u W_\lambda$ contains a unique shortest element u w.r.t. Bruhat order.

Then left multiplication by u defines

$$\begin{array}{ccccc} W_\lambda & \xrightarrow[\cong]{u\cdot -} & uW_\lambda & \hookrightarrow & W_\Theta u W_\lambda \\ \| & & \| & & \| \\ \bigsqcup W_\lambda \cap u^{-1}W_\Theta \nu & \xrightarrow{\cong} & \bigsqcup (uW_\lambda \cap W_\Theta \nu) & \hookrightarrow & \bigsqcup W_\Theta \nu \end{array}$$

- Each u determines a subset $\Theta(u,\lambda)$ of simple reflections in W_λ so that the partition $\bigsqcup W_\lambda \cap u^{-1}W_\Theta \nu$ is the same as the partition given by right cosets $W_{\lambda,\Theta(u,\lambda)}\backslash W_\lambda$.

Write $\mathcal{H}_{\Theta(u,\lambda)} = \mathbb{Z}[q,q^{-1}]\cdot(W_{\lambda,\Theta(u,\lambda)}\backslash W_\lambda)$. Then Romanov's work tells us that there is a right $\mathcal{H}_\lambda$-action on $\mathcal{H}_{\Theta(u,\lambda)}$, and we know how to write down KL basis elements in $\mathcal{H}_{\Theta(u,\lambda)}$. From the bottom row of the above diagram, each coset $D \in W_\Theta\backslash W$ gives rise to a unique coset $D|_\lambda \in W_{\lambda,\Theta(u,\lambda)}\backslash W_\lambda$. So if D and E are in the same double coset $W_\Theta u W_\lambda$, we can talk about the WKL polynomial $P^{u,\lambda}_{D|_\lambda,E|_\lambda}$ in $\mathcal{H}_{\Theta(u,\lambda)}$.

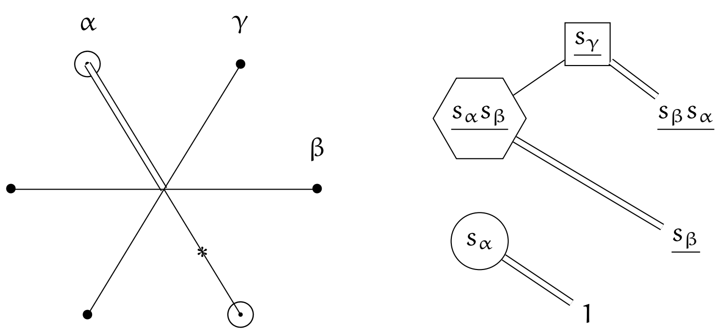
Theorem 2.3 (Non-int WKL, Z.)**.** *If λ is regular, there is a KL algorithm for $\mathcal{N}_{\lambda,\eta}$. In $\mathcal{H}_\Theta$, the following expression holds*

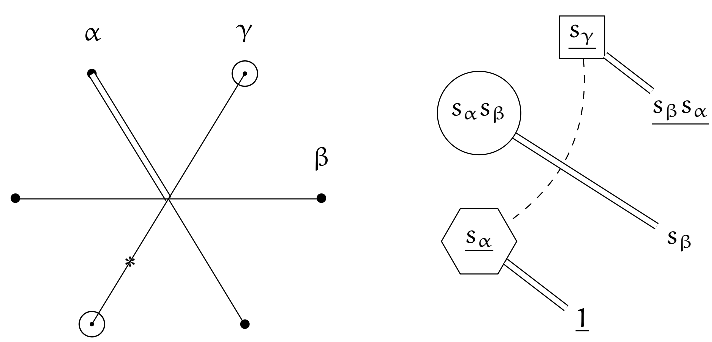
$$C(D,\lambda,\eta) = \sum_{\substack{E\in W_\Theta\backslash W_\Theta u W_\lambda \\ E|_\lambda \leqslant_{u,\lambda} D|_\lambda}} P^{u,\lambda}_{D|_\lambda,E|_\lambda}(q)T(E,\lambda,\eta)$$

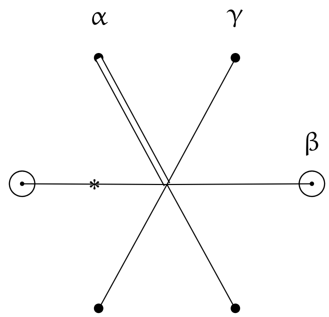
where $P^{u,\lambda}_{D|_\lambda,E|_\lambda}(q)$ are the WKL polynomials. Specializing at $q=-1$, we get the character formula for irreducibles.

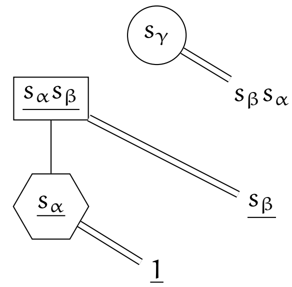
At the end, let us demonstrate how the algorithm works in the non-integral case.

Example 2.4 (A2)**.** $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{C})$, $\Theta = \{\alpha\}$, $\lambda \in W(-\frac{1}{2}\alpha)$.









A detailed description of how the algorithm runs in this example can be found in my Ph.D. thesis, Appendix A.1.