IRREDUCIBILITY CRITERION FOR STANDARD MODULES: EXAMPLES

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These notes are written for a talk at the Representation Theory Seminar at the University of Melbourne in May, 2022. This talk is a follow up to previous two talks. The notes for the previous talks can be found here.

Warning: in the previous talks (especially in examples), labelled roots (α, β, γ, etc.) are usually roots on Cartan subalgebras. In this talk, the labelled roots are all roots on the *universal* Cartan algebra.

Contents

In this talk, I will rephrase the irreducibility criterion in algebraic terms (i.e. without exponentials), demonstrate the proof of the irreducibility criterion explicitly on concrete examples (SU(2,1), SL(3,R) and SL(2,R)), and answer some questions raised in previous talks along the way.

Recall that we are fixing a complex semisimple Lie algebra $\mathfrak{g}$, an involution θ on $\mathfrak{g}$ with fixed point $\mathfrak{k}$, a semisimple complex algebraic group G with Lie algebra $\mathfrak{g}$ and a reductive subgroup K with Lie algebra $\mathfrak{k}$. $\mathfrak{h}$ denotes the universal Cartan algebra of $\mathfrak{g}$, and $\lambda \in \mathfrak{h}^*$ is fixed. We consider irreducibility of standard $\mathrm{Mod}_{coh}\,\mathcal{D}_{X,\lambda}, K)$-modules $\mathcal{I}(Q,\tau) := j_{Q*}\tau$ where $j_Q : Q \to X$ is the immersion and τ is an irreducible K-equivariant connection on Q.

1. The criterion rephrased

It was requested to rephrase the parity condition in algebraic group terms without exponentials.

Write H for the *universal Cartan group* of G, $\Phi \supset \Phi^-$ its root system and the set of negative roots. We will often pass between (co)roots/weights of H and $\mathfrak{h}$ freely without changing notations. Recall that each K-orbit Q on X determines an involution θ_Q on H by choosing a point $x \in Q$, choosing a θ-stable Cartan subgroup $C \subset B_x$, and pulling back θ on C along the restriciton map $(-)|_x : H \cong B_x/N_x \cong C$. The involution θ_Q determines different types of roots, the collections of which are denoted by

$$\Phi_{Q,\mathbb{R}}, \quad \Phi_{Q,\mathbb{C}}, \quad \Phi_{Q,NI}, \quad \Phi_{Q,CI}. \tag{1.1}$$

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Let $x \in Q$. For each $\alpha \in \Phi_{Q,\mathbb{R}}$, we have the following commutative diagram

$$\begin{tikzcd}
 & & \{-1\} \arrow[r, hook] \arrow[d, "\alpha^\vee"] \arrow[dl, dotted, bend left=0, to path={-- ([yshift=0pt]\tikztotarget)}, shift left=0] \arrow[ddl, dotted] & \mathbb{G}_m \arrow[d, "\alpha^\vee"] \\
 & & H^{\theta_Q} \arrow[r, hook] \arrow[d, "(-)|_x"] & H \arrow[d, "(-)|_x"] \\
\mathbb{G}_m & K \cap B_x \arrow[l, "\tau(x)"'] & T \arrow[l, hook'] \arrow[r, hook] & C
\end{tikzcd} \tag{1.2}$$

where the dotted arrow is the composition. The image of -1 along the dotted arrow is denoted by $m_{\alpha|x}$, i.e.

$$m_{\alpha|x} := \alpha^\vee(-1)|_x. \tag{1.3}$$

Hence $\tau(x)(m_{\alpha|x}) = \tau(x)(\alpha^\vee(-1)|_x) \in \{\pm 1\}$.

There is a subset of complex roots

$$D_-(Q) = \left\{\alpha \in \Phi^-_{Q,\mathbb{C}} \mid -\theta_Q\alpha \in \Phi^-_{Q,\mathbb{C}}\right\}. \tag{1.4}$$

On Zuckerman orbits this set is empty.

Parity Condition 1.5. *Let $\mathcal{I}(Q,\tau)$ be a standard module. Let $\alpha \in \Phi_{Q,\mathbb{R}}$ and $x \in Q$. Let A be a set of representatives of $(-\theta_Q)$-orbits in $D_-(Q)$. We say that τ satisfies the **$\mathbf{SL}_2$-parity condition with respect to** α if*

$$(-1)^{\langle \alpha^\vee,\, \lambda+\rho+\sum_{\beta\in A}\beta\rangle} \neq -\tau(x)\big(m_{\alpha|x}\big). \tag{1.6}$$

This does not depend on the choice of A. The pairing on the left side is computed on the Lie algebra level.

Irreducibility Criterion 1.7. *Let $\mathcal{I}(Q,\tau)$ be a standard module. Then $\mathcal{I}(Q,\tau)$ is irreducible if and only if both of the following conditions are satisfied:*

- *$D_-(Q) \cap \Phi_\lambda = \varnothing$, and*
- *τ satisfies $\mathbf{SL}_2$-parity condition with respect to all roots in $\Phi^-_{Q,\mathbb{R}}$.*

The plan for today is to run the proof on small examples very explicitly. Hopefully this will help people understand what is going on.

2. Example: $\mathbf{SU}(2,1)$

Recall that there are six orbits attached to two classes of Cartans:

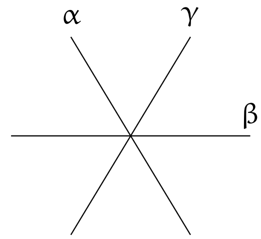
$$\begin{tikzcd}[column sep=small]
 & & O \arrow[dl, no head] \arrow[dr, no head] & & & \mathfrak{c}_{m.sp} \\
 & Q_+ \arrow[dl, no head] \arrow[dr, no head] & & Q_- \arrow[dl, no head] \arrow[dr, no head] & & \\
C_+ & & C_0 & & C_- & \mathfrak{c}_{cpt}
\end{tikzcd} \tag{2.1}$$

2.1. Types of roots determined by different orbits.

In one of the previous talks there was some confusion on how different types of roots are determined on the universal Cartan. Let us answer that here by making everything precise.

The universal Cartan algebra $\mathfrak{h}$ has the following root system where the negative roots are labelled by α, β, γ:

$\Phi :$



. (2.2)

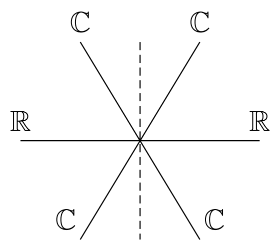
The roots α, β, γ do note depend on any point on X or on any orbit.

Let

$$\mathfrak{c}_{m.sp} = \left\{ \begin{pmatrix} a & & b \\ & -2a & \\ b & & a \end{pmatrix} \mid a, b \in \mathbb{C} \right\} \tag{2.3}$$

be a maximally split Cartan subalgebra. Then its root system is

$\Phi(\mathfrak{g}, \mathfrak{c}_{m.sp}) :$

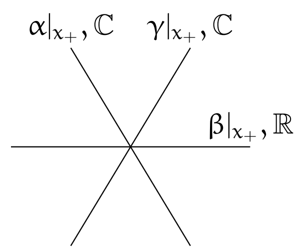


. (2.4)

These types of roots are determined by θ and does not depend on any orbit.

Now consider the orbit Q_+. If we take the Borel $\mathfrak{b}_{x_+} \supset \mathfrak{c}_{m.sp}$ by taking the roots from 11 to 3 o'clock, the corresponding point x_+ lies in Q_+. Therefore we have a restriction map $(-)|_{x_+} : \mathfrak{h}^* \to \mathfrak{c}^*_{m.sp}$ that sends

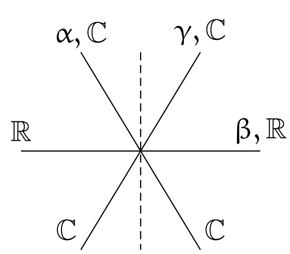
$(-)|_{x_+} : \Phi^- \mapsto$



(2.5)

Pulling back to $\mathfrak{h}^*$, we see that $\alpha, \gamma \in \Phi^-_{Q_+,\mathbb{C}}$ and $\beta \in \Phi^-_{Q_+,\mathbb{R}}$. Hence the picture on Φ determined by Q_+ is

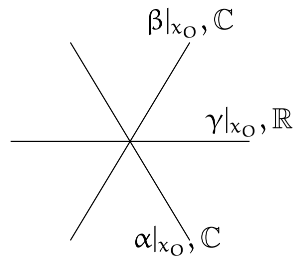
$(\Phi, \theta_{Q_+}) :$



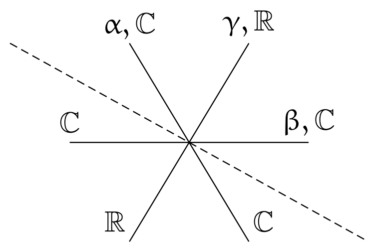
(2.6)

Now consider the orbit O. Let $\mathfrak{b}_{x_O} \supseteq \mathfrak{c}_{m.sp}$ be formed by taking the roots from 1 to 5 o'clock. Then $x_O \in O$. Since $\mathfrak{b}_{x_+}$ and $\mathfrak{b}_{x_O}$ are in relative position s_α, $(-)|_{x_O} = (s_\alpha -)|_{x_+}$. So the restriction map $(-)|_{x_O}$:

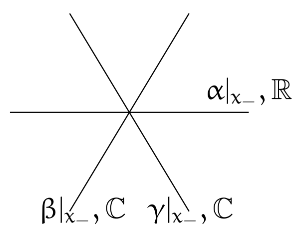
$\mathfrak{h}^* \to \mathfrak{c}^*_{\mathfrak{m}.sp}$ sends

$(-)|_{x_O} : \Phi^- \mapsto$  . (2.7)

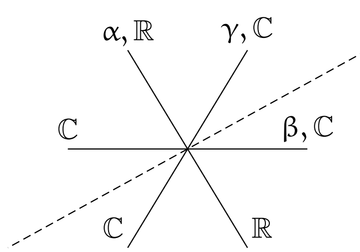
Pulling back to $\mathfrak{h}^*$, we see that

(Φ, θ_O) :  (2.8)

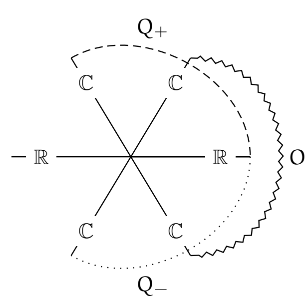
For Q_-, we can form $\mathfrak{b}_{x_-} \supset \mathfrak{c}_{\mathfrak{m}.sp}$ by taking roots from 3 to 7 o'clock. Then

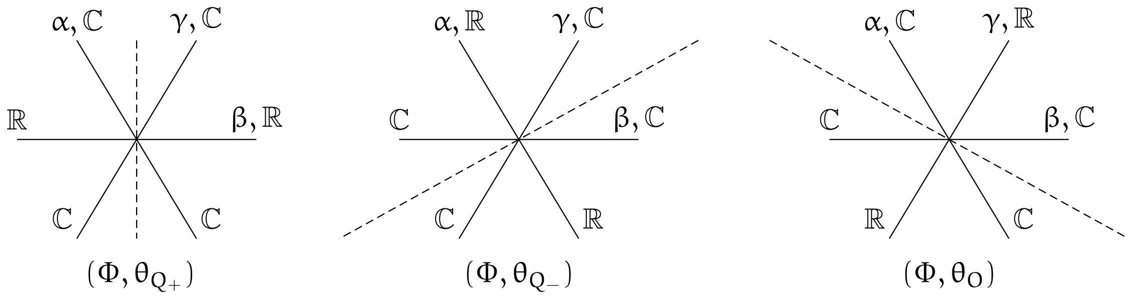
$(-)|_{x_-} : \Phi^- \mapsto$  . (2.9)

Hence Pulling back to $\mathfrak{h}^*$, we see that

(Φ, θ_{Q_-}) :  (2.10)

Putting these together, we get the picture on $\mathfrak{c}^*_{\mathfrak{m}.sp}$ from previous talks

$\Phi(\mathfrak{g}, \mathfrak{c}_{\mathfrak{m}.sp})$:  (2.11) and the following three pictures on the universal Cartan $\mathfrak{h}^*$:



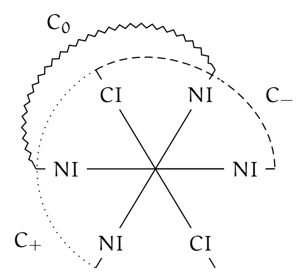
(2.12)

Using the same argument on the compact Cartan

$$\mathfrak{c}_{\text{cpt}} = \left\{ \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix} \right\}, \tag{2.13}$$

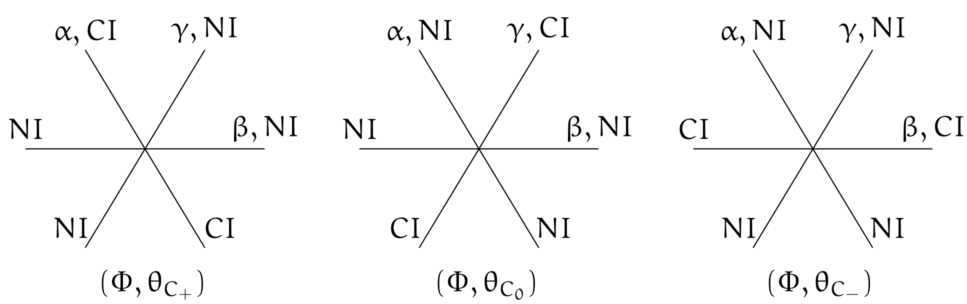
we have the picture on $\mathfrak{c}^*_{\text{cpt}}$

$\Phi(\mathfrak{g}, \mathfrak{c}_{\text{cpt}})$:



(2.14)

and the following pictures on $\mathfrak{h}^*$:



(2.15)

2.2. **Standard modules on Zuckerman orbits.** Consider the standard module $\mathcal{I}(Q_+, \tau_+)$ on Q_+ (since the K-stabilizer of $x_+ \in Q_+$ is connected, there is a unique connection τ_+ on Q_+ compatible with λ). According to the proof, we should project to the partial flag variety corresponding to Q_+-real simple roots. In our situation there is only one such root, namely β. So we want to project along $\pi_\beta : X \to X_{s_\beta}$.

Question 2.16. *What is the image of each orbit under π_β?*

Given an orbit $Q \subset X$, the image $\pi_\beta(Q)$ is a K-orbit in X_{s_β}. The preimage $X_{\pi_\beta(Q)} := \pi_\beta^{-1}(\pi_\beta(Q))$ is a union of finitely many orbits.

By K-equivariance, the orbit structure in $X_{\pi_\beta(Q)}$ can be detected on a fiber: if $y \in \pi_\beta(Q)$, then the fiber X_y is the flag variety of $[\mathfrak{p}_y, \mathfrak{p}_y]$ with the action of $K_y := \operatorname{im}(K \cap P_y \to P_y \to (P_y/\operatorname{rad} P_y)/Z(P_y/\operatorname{rad} P_y))$.

By K-equivariance, the K-orbit structure on $X_{\pi_\beta(Q)}$ is the same as the K_y-orbit structure on X_y.

$$\begin{tikzcd}
 & Q' \arrow[d] & Q' \cap \pi_\beta^{-1}(y) \arrow[l] \arrow[d] \\
X \arrow[d, "\pi_\beta"] & X_{\pi_\beta(Q)} \arrow[l] \arrow[d] & X_y \arrow[l] \arrow[d] \\
X_\beta & \pi_\beta(Q) \arrow[l] & \{y\} \arrow[l]
\end{tikzcd} . \tag{2.17}$$

Remark 2.18. In fact, as we have seen last time, descent gives a equivalence of categories

$$\mathrm{Mod}_{coh}(\mathcal{D}_{X_{\pi_\beta(Q)},\lambda}, K) \cong \mathrm{Mod}_{coh}(\mathcal{D}_{X_y,\lambda}, K_y). \tag{2.19}$$

On the other hand, $X_y \cong \mathbb{P}^1$, and there are only three possibilities for K_y° (identity component) if K_y has finitely many orbits on X_y: K_y° is isomorphic to either $\mathbf{SL}(2,\mathbb{C})$, or $\mathbf{PSL}(2,\mathbb{C})$, or covers of $\{(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix})\} \subset \mathbf{PSL}(2,\mathbb{C})$, or $\{(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix})\} \subset \mathbf{PSL}(2,\mathbb{C})$, or $\mathbb{C}^*$. If you try to calculate K_y° by its Lie algebra and determine which case we are in, you will end up looking at the root β:

Lemma 2.20 ([Hec+, 6.5])**.** *Suppose $\beta \in \mathfrak{h}^*$ is a simple root and Q is a K-orbit.*

β	K_y°	$Q \cap X_y \subset \mathbb{P}^1$	Q
Q-*CI*	$\mathbf{SL}(2,\mathbb{C})$ *or* $\mathbf{PSL}(2,\mathbb{C})$	$Q \cap X_y = X_y$	$Q = X_{\pi_\beta(Q)}$
Q-*NI*	$\mathbb{C}^*$	$Q \cap X_y \subseteq \{0,\infty\}$	Q *is closed in* $X_{\pi_\beta(Q)}$
Q-$\mathbb{R}$	$\mathbb{C}^*$	$Q \cap X_y = \mathbb{C}^*$	Q *is open in* $X_{\pi_\beta(Q)}$
Q-$\mathbb{C}$ *and* $\beta \in D_-(Q)$	*has 1-dim'l unip. rad.*	$Q \cap X_y = \mathbb{P}^1 - \{0\}$	Q *is open in* $X_{\pi_\beta(Q)}$
Q-$\mathbb{C}$ *and* $\beta \notin D_-(Q)$	*has 1-dim'l unip. rad.*	$Q \cap X_y = \{0\}$	Q *is the closed orbit in* $X_{\pi_\beta(Q)}$

Example 2.21 (Prototypical examples)**.** The above table is proven based on the following $\mathfrak{sl}_2$-examples. Let $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$ and let $\pm\beta$ be the roots.

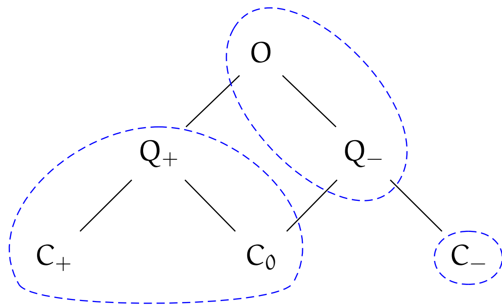
β	G_0	$(\mathfrak{g}, K)$	Q
Q-CI	$\mathbf{SU}(2)$	$(\mathfrak{sl}(2,\mathbb{C}), \mathbf{SL}(2,\mathbb{C}))$	$\mathbb{P}^1$
Q-NI	$\mathbf{SL}(2,\mathbb{R})$	$(\mathfrak{sl}(2,\mathbb{C}), \mathbf{SO}(2,\mathbb{C}))$	$\{0\}$ or $\{\infty\}$
Q-$\mathbb{R}$	$\mathbf{SL}(2,\mathbb{R})$	$(\mathfrak{sl}(2,\mathbb{C}), \mathbf{SO}(2,\mathbb{C}))$	$\mathbb{C}^*$
Q-$\mathbb{C}$, $\beta \in D_-(Q)$	"$\mathbf{SL}(2,\mathbb{C})$"	$(\mathfrak{sl}(2,\mathbb{C}), \{(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix})\})$	$\mathbb{P}^1 - \{0\}$
Q-$\mathbb{C}$, $\beta \notin D_-(Q)$	"$\mathbf{SL}(2,\mathbb{C})$"	$(\mathfrak{sl}(2,\mathbb{C}), \{(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix})\})$	$\{0\}$

Remark 2.22. A question raised last time was: why can we always find a $\mathbb{P}^1$-slice in X transversal to a closed subset S in X? This lemma answers the question when S is the closure of an orbit Q: we can take a simple root β that is Q-NI or Q-$\mathbb{C}$ and $\beta \notin D_-(Q)$, and take the $\mathbb{P}^1$-slice to be X_y.

Back to $\mathbf{SU}(2,1)$. Take $y = \pi_\beta(x_+) \in \pi_\beta(Q_+)$. Then $Q_+ \cap X_y = \mathbb{C}^*$ is the open $\mathbb{C}^*$-orbit. So any other orbit Q' in $X_{\pi_\beta(Q_+)}$ must be 1-dimensional (hence must be a closed orbit), lies in $\overline{Q_+}$, and β must be Q'-NI. Looking back at (2.15), Q' could be either C_+ or C_0. If one of them is not in $X_{\pi_\beta(Q_+)}$, then there must be another 2-dimensional orbit Q'' and β is Q''-$\mathbb{R}$. But the only other 2-dimensional orbit is Q_- and β is Q_--$\mathbb{C}$. Therefore both C_+ and C_0 are in $X_{\pi_\beta(Q_+)}$. The three orbits $Q_+ \cap X_y$, $C_+ \cap X_y$ and $C_0 \cap X_y$ can be identified with $\mathbb{C}^*$, $\{0\}$ and $\{\infty\}$ on $\mathbb{P}^1$.

Let's look at other orbits. For C_-, β is C_--CI, so $C_- = X_{\pi_\beta(C_-)}$ and $\pi_\beta(C_-)$ is a point. For Q_-, β is Q_--$\mathbb{C}$ and $\beta \notin D_-(Q_-) = \varnothing$. So there is a $\mathbb{P}^1$-slice (denoted by, say, $\mathbb{P}^1_\beta$) so that $Q_- \cap \mathbb{P}^1_\beta = \{0\}$. For O, β is O-$\mathbb{C}$ and $\beta \in D_-(O)$, so $O \cap \mathbb{P}^1_\beta = \mathbb{P}^1 - \{0\}$.

Therefore we have the following picture, where orbits in the same circle are mapped onto the same orbit:



(2.23)

$O \cup Q_-$ is mapped to the 2-dimensional open orbit in X_{s_β}; $Q_+ \cup C_+ \cup C_0$ are mapped to a 1-dimensional closed orbit; C_- is mapped onto a point.

Let's come back to irreducibility of $\mathcal{I}(Q_+, \tau_+)$. First, $X_{\pi_\beta(Q)} = Q_+ \cup C_+ \cup C_0$ is closed in X (because $\pi_\beta(Q)$ is closed in X_{s_β}). So by Kashiwara's theorem there is an equivalence

$$\underbrace{\mathrm{Mod}_{coh}(\mathcal{D}_{X,\lambda}, K)_{Q_+\cup C_+\cup C_0}}_{\text{modules supported in } Q_+\cup C_+\cup C_0} \cong \mathrm{Mod}_{coh}(\mathcal{D}_{Q_+\cup C_+\cup C_0,\lambda}, K) \tag{2.24}$$

and $\mathcal{I}(Q_+, \tau_+)$ is irreducible iff. it is irreducible in the second category. Then, if $y = \pi_\beta(x_+) \in \pi_\beta(Q_+)$, by descent 2.18, taking $*$-pullback to X_y is an equivalence

$$\mathrm{Mod}_{coh}(\mathcal{D}_{Q_+\cup C_+\cup C_0,\lambda}, K) \cong \mathrm{Mod}_{coh}(\mathcal{D}_{X_y,\lambda}, K_y) \cong \mathrm{Mod}_{coh}(\mathcal{D}_{\mathbb{P}^1,\lambda}, K_y) \tag{2.25}$$

By base change, $\mathcal{I}(Q_+, \tau_+)|_{X_y}$ is a standard module $\mathcal{I}(\mathbb{C}^*, \tau_+|_{Q_+\cap X_y})$ on $\mathbb{C}^*$. The representation $\tau_+|_{Q_+\cap X_y}(x_+)$ defining the connection is the restriction of $\tau_+(x_+)$ to the K_y-stabilizer of x_+ which is $\{1, m_{\beta|x_+}\}$. We have seen last time that $\mathcal{I}(\mathbb{C}^*, \tau_\varepsilon)$ is irreducible if and only if

$$(-1)^{\langle \beta^\vee,\, \lambda+\rho\rangle} \neq -\tau_+|_{Q_+\cap X_y}(x_+)(m_{\beta|x_+}). \tag{2.26}$$

This is the same as the parity condition for τ_+. Therefore,

$$\begin{aligned} & \mathcal{I}(Q_+, \tau_+) \text{ irreducible} \\ \iff & \mathcal{I}(Q_+, \tau_+)|_{X_y} = \mathcal{I}(\mathbb{C}^*, \tau_\varepsilon) \text{ is irreducible} \\ \iff & \tau_+|_{Q_+\cap X_y} \text{ satisfies parity condition for } \beta \\ \iff & \tau_+ \text{ satisfie parity condition for } \beta. \end{aligned}$$

A similar argument works for Q_-.

2.3. **Standard module on the open orbit.** Let's consider a standard module $\mathcal{I}(O, \tau_O)$ on O (again there is a unique connection τ_O compatible with λ because of connectedness of $K \cap B_{x_O}$). In this case γ is the only real root and $D_-(O) = \{\alpha, \beta\}$. Take $A = \{\alpha\}$. The parity condition for τ_O reads

$$(-1)^{\langle \gamma^\vee,\, \lambda+\rho+\alpha\rangle} \neq -\tau_O(x_O)(m_{\gamma|x_O}). \tag{2.27}$$

This is supposed to be a "translation" of the parity condition for a Zuckerman orbit (which can be taken to be Q_+). Let τ_+ be the unique connection on Q_+ compatible with $s_\alpha\lambda$.

Claim 2.28. The $\mathcal{D}_{X,\lambda}$-module $\mathcal{I}(O, \tau_O)$ satisfies the parity condition for $\gamma \iff$ the $\mathcal{D}_{X,s_\alpha\lambda}$-module $\mathcal{I}(Q_+, \tau_+)$ satisfies the parity condition for β. In other words,

$$(-1)^{\langle \gamma^\vee,\, \lambda+\rho+\alpha\rangle} \neq -\tau_O(x_O)(m_{\gamma|x_O}) \tag{2.29}$$

$$\iff (-1)^{\langle \beta^\vee,\, s_\alpha\lambda+\rho\rangle} \neq -\tau_+(x_+)(m_{\beta|x_+}). \tag{2.30}$$

What is the relationship between Q_+ and O and between τ_+ and τ_O? From (2.11) we see that $\mathfrak{b}_{x_+}$ and $\mathfrak{b}_{x_O}$ are in relative position α. So we have the following commutative diagram

$$\begin{tikzcd} O \arrow[d, hook, "j_O"'] & p_2^{-1}(Q_+) \arrow[l, "\cong"'] \arrow[r, "p_2"] \arrow[d, hook] & Q_+ \arrow[d, hook, "j_{Q_+}"] \\ X & Z_{s_\alpha} \arrow[l, "p_1"'] \arrow[r, "p_2"] & X \end{tikzcd} \tag{2.31}$$

which was used to show

$$\mathbb{I}_{s_\alpha}\mathcal{I}(Q_+,\tau_+) = \mathcal{I}(O,\tau_O) \tag{2.32}$$

where Z_{s_α} is the G-orbit in $X \times X$ labelled by s_α. Inverting the isomorphism in the diagram, we get a K-equivariant submersion

$$O \twoheadrightarrow Q_+ \tag{2.33}$$

corresponding to the inclusion

$$\left\{\begin{pmatrix} z & & \\ & z^{-2} & \\ & & z \end{pmatrix}\right\} = K \cap B_{x_O} \hookrightarrow K \cap B_{x_+} = \left\{\begin{pmatrix} z & bz & \\ & z^{-2} & \\ & & z \end{pmatrix}\right\}. \tag{2.34}$$

The connection τ_O is the $*$-pullback of τ_+ along this submersion, and the representation $\tau_O(x_O)$ is given as the restriction of $\tau_+(x_+)$, i.e.

$$\tau_O(x_O) = \tau_+(x_+)|_{K\cap B_{x_O}}. \tag{2.35}$$

Bringing the definitions of $m_{\beta|x_+}$ and $m_{\gamma|x_O}$ into our picture, we have a commutative diagram:

$$\begin{tikzcd}[column sep=small] & & & \{-1\} \arrow[rr, hook] \arrow[dr, equal] \arrow[dd, "\gamma^\vee"'] \arrow[ddddlll, squiggly] & & \mathbb{G}_m \arrow[dr, equal] \arrow[dd, "\gamma^\vee"'] & \\ & & & & \{-1\} \arrow[rr, hook] \arrow[dd, "\beta^\vee"] \arrow[ddddlll, dashed] & & \mathbb{G}_m \arrow[dd, "\beta^\vee"] \\ & & & H^{\theta_O} \arrow[rr, hook] \arrow[dr, "s_\alpha"] \arrow[dd, "(-)|_{x_O}"'] & & H \arrow[dr, "s_\alpha"] \arrow[dd, "(-)|_{x_O}"'] & \\ & & & & H^{\theta_{Q_+}} \arrow[rr, hook] \arrow[dd, "(-)|_{x_+}"] & & H \arrow[dd, "(-)|_{x_+}"] \\ \mathbb{G}_m \arrow[dr, equal] & K \cap B_{x_O} \arrow[l, "\tau_O(x_O)"'] \arrow[dr, hook] & & T_{m.sp} \arrow[ll, hook'] \arrow[rr, hook] \arrow[dr, equal] & & C_{m.sp} \arrow[dr, equal] & \\ & \mathbb{G}_m & K \cap B_{x_+} \arrow[l, "\tau_+(x_+)"'] & & T_{m.sp} \arrow[ll, hook'] \arrow[rr, hook] & & C_{m.sp} \end{tikzcd} . \tag{2.36}$$

$\tau_O(x_O)(m_{\gamma|x_O}) = \tau_O(x_O)(\gamma^\vee(-1)|_{x_O})$ is the image along the squiggly arrow, and $\tau_+(x_+)(m_{\beta|x_+}) = \tau_+(x_+)(\beta^\vee(-1)|_{x_+})$ is the image of the dashed arrow. Therefore

$$\tau_O(x_O)(m_{\gamma|x_O}) = \tau_+(x_+)(m_{\beta|x_+}). \tag{2.37}$$

Thus the claim reduces to

$$(-1)^{\langle\gamma^\vee,\,\lambda+\rho+\alpha\rangle} = (-1)^{\langle\beta^\vee,\,s_\alpha\lambda+\rho\rangle} \tag{2.38}$$

which can be verified easily.

Let's come back to irreducibility of $\mathcal{I}(O,\tau_O)$. The criterion involves an additional condition: $D_-(O) \cap \Phi_\lambda = \varnothing$, which has to do with intertwining functor $\mathbb{I}_{s_\alpha}$.

Since $\mathbb{I}_{s_\alpha}\mathcal{I}(Q_+,\tau_+) = \mathcal{I}(O,\tau_O)$, if $\mathbb{I}_{s_\alpha}$ is an equivalence of categories, then

$$
\begin{aligned}
& \mathcal{I}(O,\tau_O) \text{ is irreducible} \\
\iff\ & \mathcal{I}(Q_+,\tau_+) \text{ is irreducible} \\
\iff\ & \tau_+ \text{ satisfies parity condition for } \beta \\
\iff\ & \tau_O \text{ satisfies parity condition for } \gamma.
\end{aligned}
$$

We know $\mathbb{I}_{s_\alpha}$ is an equivalence of categories if and only if $\alpha \notin \Phi_\lambda$, i.e. α not integral. On the other hand, the compatibility of τ_+ and λ forces $\alpha + \theta_O\alpha = \alpha - \beta$ to be integral. So α not integral iff. α, β both not integral iff. $D_-(O) \cap \Phi_\lambda = \{\alpha, \beta\} \cap \Phi_\lambda = \varnothing$. We have shown:

$$\text{“}D_-(O) \cap \Phi_\lambda = \varnothing\text{” + “}\tau_O \text{ satisfies parity condition for } \gamma\text{”} \implies \text{“}\mathcal{I}(O,\tau_O) \text{ irreducible”}.$$

What if $D_-(O) \cap \Phi_\lambda \neq \varnothing$? Then α or β is integral[1]. Say β is integral. Then if we look at the $\mathbb{P}^1$-slice $\mathbb{P}^1_\beta$ through O and Q_-, the standard module $\mathcal{I}(\mathbb{P}^1_\beta - \{0\}, \lambda) = \mathcal{I}(O,\tau_O)|_{\mathbb{P}^1_\beta}$ is reducible (the submodule is a line bundle on $\mathbb{P}^1_\beta$). By descent, $\mathcal{I}(O,\tau_O)|_{O\cup Q_-}$ is reducible, and hence $\mathcal{I}(O,\tau_O)$ must be reducible. We have therefore shown:

$$\text{“}D_-(O) \cap \Phi_\lambda = \varnothing\text{” + “}\tau_O \text{ satisfies parity condition for } \gamma\text{”} \iff \text{“}\mathcal{I}(O,\tau_O) \text{ irreducible”}.$$

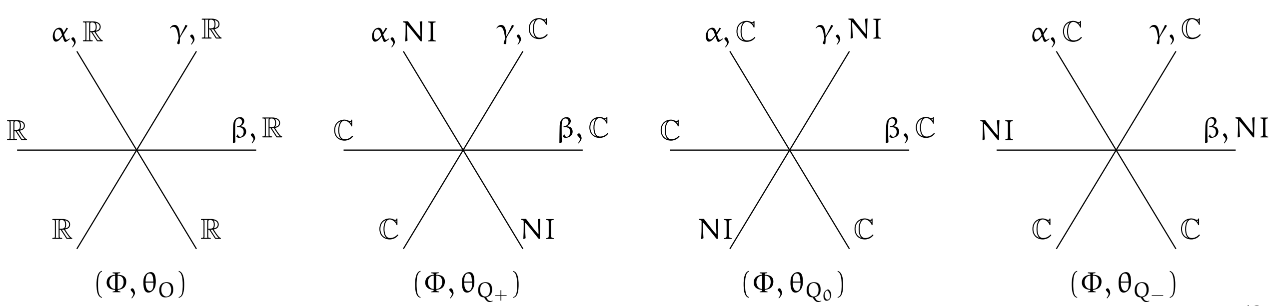
3. Example: $\mathbf{SL}(3,\mathbb{R})$

The only phenomenon not present in $\mathbf{SU}(2,1)$ is a standard module on a large open orbit for a split pair. So let's look at th open orbit in $\mathbf{SL}(3,\mathbb{R})$. Recall that there are four orbits:

$$
\begin{tikzcd}[column sep=small]
 & O \arrow[dl, dash] \arrow[dr, dash] & & \mathfrak{c}_{sp} \\
Q_+ \arrow[dr, dash] & & Q_- \arrow[dl, dash] & \\
 & Q_0 & & \mathfrak{c}_{m.cpt}
\end{tikzcd}
, \tag{3.1}
$$

3.1. Types of roots determined by different orbits.

Repeating what we did in §2.1, we obtain the following pictures on $\mathfrak{h}^*$:



(3.2)

3.2. Standard modules on the open orbit.

Consider the open orbit O. As we discussed last time, there are four irreducible connections on O compatible with any λ. Let τ be one of them.

We first want to obtain a necessary condition for irreducibility of $\mathcal{I}(O,\tau)$. So suppose $\mathcal{I}(O,\tau)$ is irreducible. We want to show that τ satisfies parity condition for all (real) roots. The idea is to reduce to $\mathfrak{sl}_2$ case by pulling back to $\mathbb{P}^1$-slices.

Consider first the simple root α. We want to pullback to a $\mathbb{P}^1_\alpha$-slice through O. Since α is O-real, any other orbit Q meeting $\mathbb{P}^1_\alpha$ has α as a Q-NI root. The only such orbit is Q_+. So $X_{\pi_\alpha(O)} = O \cup Q_+$, $O \cap \mathbb{P}^1_\alpha = \mathbb{C}^*$, and $Q_+ \cap \mathbb{P}^1_\alpha = \{0\} \cup \{\infty\}$[2]. Therefore

[1] In fact they are both integral because $\alpha - \beta$ is integral

[2] The intersection consists of two orbits because K_y is disconnected.

$$\begin{aligned}
\mathcal{I}(O,\tau) \text{ is irreducible} &\implies \mathcal{I}(O,\tau)|_{O\cup Q_+} \text{ is irreducible}\\
\text{(by descent)} &\iff \mathcal{I}(O,\tau)|_{\mathbb{P}^1_\alpha} \text{ is irreducible}\\
&\iff \tau|_{O\cap\mathbb{P}^1_\alpha} \text{ satisfies parity condition for } \alpha\\
&\iff \tau \text{ satisfies parity condition for } \alpha.
\end{aligned}$$

A similar argument works for the other simple real root β.

$\mathcal{I}(O,\tau)$ is irreducible $\implies$ τ satisfies parity condition for β.

It remains to look at the real non-simple root γ. The idea is to apply s_α so that $s_\alpha\gamma = \beta$ becomes a simple real root. On the module level, this amounts to applying the intertwining functor $\mathbb{I}_{s_\alpha}$.

Lemma 3.3. *Suppose τ satisfies parity condition for a simple real root α.*

(1) *There exists a connection on O such that*

$$\mathbb{I}_{s_\alpha}\mathcal{I}(O,\tau) = \mathcal{I}(O,\tau_{s_\alpha}). \tag{3.4}$$

(2) *τ satisfies parity condition for γ if and only if τ_{s_α} satisfies parity condition for $s_\alpha\gamma = \beta$.*

(3) *If $\alpha^\vee(\lambda) \in \mathbb{Z}$, then*

$$\mathcal{I}(O,\tau_{s_\alpha}) = \mathcal{I}(O,\tau)(-\alpha^\vee(\lambda)\alpha). \tag{3.5}$$

Assuming the lemma, we have two cases:

(a) $\alpha^\vee(\lambda) \notin \mathbb{Z}$. In this case $\mathbb{I}_{s_\alpha}$ is an equivalence of categories, so

$$\begin{aligned}
\mathcal{I}(O,\tau) \text{ irreducible} &\iff \mathcal{I}(O,\tau_{s_\alpha}) \text{ irreducible}\\
\text{(since } \beta \text{ is simple)} &\implies \tau_{s_\alpha} \text{ satisfies parity condition for } \beta = s_\alpha\gamma\\
&\iff \tau \text{ satisfies parity condition for } \gamma.
\end{aligned}$$

(b) $\alpha^\vee(\lambda) \in \mathbb{Z}$. Since twisting is also an equivalence of categories, the previous argument goes through.

Therefore, we have shown

$\mathcal{I}(O,\tau)$ is irreducible $\implies$ τ satisfies parity condition for all (real) roots.

Let's consider the converse: suppose τ satisfies parity condition for all roots and $\mathcal{I}(O,\tau)$ is reducible. Then there is a proper quotient $\mathcal{I}(O,\tau) \twoheadrightarrow \mathcal{K}$ with irreducible support (so $\operatorname{Supp}\mathcal{K} = \overline{Q}$ for some orbit Q). Then there are two cases:

(a) The good case is when $\operatorname{Supp}\mathcal{K}$ has codimension 1. Then $\operatorname{Supp}\mathcal{K}$ is the closure of either Q_+ or Q_-. Say it's Q_+. Then restricting to the $\mathbb{P}^1_\alpha$-slice we found before produces a surjection

$$\mathcal{I}(\mathbb{C}^*,\tau|_{O\cap\mathbb{P}^1_\alpha}) = \mathcal{I}(O,\tau)|_{\mathbb{P}^1_\alpha} \twoheadrightarrow \mathcal{K}|_{\mathbb{P}^1_\alpha} \neq 0 \tag{3.6}$$

where $\mathcal{K}|_{\mathbb{P}^1_\alpha}$ is supported in $\{0,\infty\}$. So $\tau|_{O\cap\mathbb{P}^1_\alpha}$ and hence τ must fail the parity condition for α. This is a contradiction.

(b) The bad case is when $\operatorname{Supp}\mathcal{K}$ has codimension > 1. So $\operatorname{Supp}\mathcal{K} = Q_0$ and $\mathcal{K}$ can be taken to be $\mathcal{I}(Q_0,\tau_0)$. The idea is to reduce to the good case by enlarging $\mathcal{I}(Q_0,\tau_0)$. We know how this can be done: intertwining functor for complex reflections.

More specifically, we take the simple Q_0-complex root β. Then

$$\mathbb{I}_{s_\beta}\mathcal{I}(Q_0,\tau_0) = \mathcal{I}(Q_+,\tau_+) \tag{3.7}$$

for some τ_+. So we obtain, by right-exactness of $H^0\mathbb{I}_{s_\beta}$, the surjections

$$\begin{array}{ccc}
H^0\mathbb{I}_{s_\beta}\mathcal{I}(O,\tau) & \twoheadrightarrow & H^0\mathbb{I}_{s_\beta}\mathcal{I}(Q_0,\tau_0)\\
\| & & \|\\
\mathcal{I}(O,\tau_{s_\beta}) & \twoheadrightarrow & \mathcal{I}(Q_+,\tau_+)
\end{array}. \tag{3.8}$$

By case (a), τ_{s_β} must fail parity condition for $\alpha = s_\beta\gamma$. So τ must fail parity condition for γ, giving a contradiction.

We have thus proved:

$\mathcal{I}(O,\tau)$ is irreducible $\iff$ τ satisfies parity condition for β

modulo the lemma.

As we have probably guessed, the lemma is proven by taking $\mathbb{P}^1_\alpha$-slice and reduce to explicit $\mathfrak{sl}(2,\mathbb{C})$-calculation. I'll give you the general steps and leave the details to the audience as an exercise:

(a) Show that for a standard module on O, $(\mathbb{I}_{s_\alpha}\mathcal{I})|_{\mathbb{P}^1_\alpha} = \mathrm{I}(\mathcal{I}|_{\mathbb{P}^1_\alpha})$, where I is the intertwining functor on $\mathbb{P}^1$.
(b) Show that Lemma(1) is true on $\mathbb{P}^1$.
(c) Show that (a)+(b) proves Lemma(1).
(d) Lemma(2) is proven in the same way as 2.28.
(e) Lemma(3) is proven by explicitly tracing through step (a).

References

[Hec+] H. Hecht et al. *Localization and standard modules for real semisimple Lie groups II: Irreducibility, vanishing theorems and classification*. Draft version.