COMPARISON OF GEOMETRIC AND LANGLANDS CLASSIFICATIONS

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Contents

Let $G = \mathrm{Int}(\mathfrak{g})$. Let $\varphi : K \to \mathrm{Int}(\mathfrak{g})$ be morphism. Let $(\mathfrak{g}_0, K_0)$ and $(\mathfrak{g}, K)$ be as usual. Assume K is connected. Let σ denote the Cartan involution on $\mathfrak{g}$ compatible with K. Let X denote the flag variety of $\mathfrak{g}$. In these notes we compare two classifications of irreducible admissible $(\mathfrak{g}, K)$-modules: the geometric classification due to Beilinson-Bernstein and Knapp-Zuckerman's version of Langlands classification [KZ77].

Both geometric classification and Langlands classification realize irreducible objects as unique irreducible submodules of certain standard modules. Therefore, to obtain a comparison, it is enough to identify the standard modules from the two sides.

These talks are based on communication with Dragan Milicic, and I would like to thank him for these discussions. Any mistakes in these notes are mine.

1. Translation to Langlands orbits

1.1. **Langlands orbits and real parabolics.** Recall that associated to each K-orbit Q is a K-conjugacy class of Cartan subalgebra:

$$\{K\text{-orbits in } X\} \to \left\{ \begin{matrix} K\text{-conjugacy class of} \\ \text{Cartan subalgebras in } \mathfrak{g} \end{matrix} \right\} \tag{1.1}$$

$$K \cdot x \mapsto \mathrm{Ad}\, K \cdot \mathfrak{c} \tag{1.2}$$

where $\mathfrak{c}$ is a σ-stable Cartan subalgebra contained in $\mathfrak{b}_x$. The preimage over $\mathrm{Ad}\, K \cdot \mathfrak{c}$ is parametrized by W_K-conjugacy classes of choices of positive roots in $R(\mathfrak{g}, \mathfrak{c})$. That is, after fixing $\mathfrak{c}$, we have a bijection

$$\{K\text{-orbits attached to } \mathfrak{c}\} \xrightarrow{\sim} \left\{ \begin{matrix} W_K\text{-conjugacy classes of} \\ \text{choices of positive roots in } R(\mathfrak{g}, \mathfrak{c}) \end{matrix} \right\}. \tag{1.3}$$

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Definition 1.4. A **Langlands orbit attached to** $\mathfrak{c}$ is an orbit attached to $\mathfrak{c}$ having maximal possible dimension. A **Zuckerman orbit attached to** $\mathfrak{c}$ is one of minimal possible dimension.

The upshot is that, from a Langlands orbit, we can obtain a cuspidal parabolic which necessarily comes from a real parabolic of $\mathfrak{g}_0$. To achieve this, we need to characterize Langlands orbits in terms of root systems in terms of the bijection (1.3).

Let Q be any orbit attached to $\mathfrak{c} = \mathfrak{t} \oplus \mathfrak{a}$, with $x \in Q$ so that $\mathfrak{b}_x \supseteq \mathfrak{c}$. Let S_x be the stabilizer of x in K. Then $Q \cong K/S_x$ and

$$\dim Q = \dim K/S_x = \dim \mathfrak{k} - \dim \mathfrak{s}_x. \tag{1.5}$$

We know that

- $\mathfrak{k}$ is the span of $\mathfrak{t}$, $\mathfrak{g}_\alpha$'s for α compact imaginary, and $(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{\sigma\alpha})^\sigma$ for α real and complex.
- $\mathfrak{b}_x$ is the span of $\mathfrak{t} \oplus \mathfrak{a}$, and $\mathfrak{g}_\alpha$'s for α positive.

Hence

- $\mathfrak{s}_x = \mathfrak{k} \cap \mathfrak{b}_x$ is the span of $\mathfrak{t}$, $\mathfrak{g}_\alpha$'s for α compact imaginary, and $(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{\sigma\alpha})^\sigma$ for α complex positive so that $\sigma\alpha$ is also positive.

Following the notation in [Hec+], write $R_{Q,CI}$, $R_{Q,NI}$, $R_{Q,I}$, $R_{Q,\mathbb{R}}$, $R_{Q,\mathbb{C}}$ for the set of compact imaginary, noncompact imaginary, imaginary, real, complex roots in $R = R(\mathfrak{g}, \mathfrak{c})$, respectively (so in fact they depend only on $\mathfrak{c}$, not on Q). Write R_Q^+ for the set of positive roots determined by $\mathfrak{b}_x$ (this does depend on Q). Write

$$D_+(Q) = \{\alpha \in R_{Q,\mathbb{C}}^+ \mid \sigma\alpha \in R_Q^+\}. \tag{1.6}$$

Then

$$\begin{aligned} \dim Q &= \dim \mathfrak{k} - \dim \mathfrak{s}_x & (1.7)\\ &= \big(\dim \mathfrak{t} + |R_{Q,CI}| + \frac{1}{2}|R_{Q,\mathbb{R}}| + \frac{1}{2}|R_{Q,\mathbb{C}}|\big) - \big(\dim \mathfrak{t} + |R_{Q,CI}^+| + \frac{1}{2}|D_+(Q)|\big) & (1.8)\\ &= \frac{1}{2}\big(|R_{Q,CI}| + |R_{Q,\mathbb{R}}| + |R_{Q,\mathbb{C}}| - |D_+(Q)|\big). & (1.9) \end{aligned}$$

Therefore a Langlands orbit Q^L corresponds to a choice of positive roots $R_{Q^L}^+$ such that $|D_+(Q^L)|$ is minimal.

Lemma 1.10 ([Hec+, 5.1, 5.10])**.** *For any σ-stable Cartan subalgebra $\mathfrak{c}$, there exists a choice of positive roots $R_{Q^L}^+$ such that $D_+(Q^L) = \varnothing$. Moreover, for any such choice, $R_{Q^L}^+ \cup R_{Q^L,I}$ is a parabolic set of roots in R.*

Sketch of proof. First choose a positive direction in the subspace $\mathfrak{c}^*|_{\sigma=-1}$. Then choose a positive root system of R such that the positive direction is "aligned" with the positive direction of $\mathfrak{c}^*|_{\sigma=-1}$.

Any complex root will have nonzero component in $\mathfrak{c}^*|_{\sigma=-1}$ (along $\mathfrak{c}^*|_{\sigma=1}$). This component will be negated when we apply σ, and the other component (the component in $\mathfrak{c}^*|_{\sigma=1}$) will be fixed. The resulting root will be negative. Hence no complex root will belong to D_+.

After such R_Q^+ is chosen, imaginary roots are precisely the positive roots that stay inside R_Q^+ under the action of σ. So a sum of two positive roots is (positive) imaginary if and only if the σ-image of both roots are positive, if and only if both roots are imaginary. ∎

Let $\mathfrak{p} \supseteq \mathfrak{b} = \mathfrak{b}_x \supseteq \mathfrak{c}$ be a parabolic and a Borel corresponding to $R_{Q^L}^+ \cup R_{Q^L,I}$ and $R_{Q^L}^+$, respectively. Let σ' denote the anti-involution of $\mathfrak{g}$ defining $\mathfrak{g}_0$.

Lemma 1.11. *Suppose $\mathfrak{c}$ is also stable under σ'. Then $\mathfrak{p}$ is the complexification of a parabolic subalgebra $\mathfrak{p}_0$ of $\mathfrak{g}_0$.*

Proof. It is enough to show that $\mathfrak{p}$ is σ'-stable. Since σ' acts on the roots by $-\sigma$, $\sigma'\mathfrak{g}_\alpha = \mathfrak{g}_{\sigma'\alpha} = \mathfrak{g}_{-\sigma\alpha}$. If α is positive imaginary, $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} = \mathfrak{g}_\alpha \oplus \sigma'\mathfrak{g}_\alpha$ is σ'-stable; if α is positive real, $\mathfrak{g}_\alpha = \sigma'\mathfrak{g}_\alpha$ is σ'-stable; if α is positive complex, $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\sigma\alpha} = \mathfrak{g}_\alpha \oplus \sigma'\mathfrak{g}_\alpha$ is also σ'-stable. Since $\mathfrak{p}$ is the span of $\mathfrak{c}$ and the above σ'-stable subspaces, $\mathfrak{p}$ is σ'-stable. ∎

Such a parabolic $\mathfrak{p}$ is said to be cuspidal (by, I think, Harish-Chandra) or σ-split. There are analogous statements for Zuckerman orbits Q^Z: those correspond to sets of positive roots $R^+_{Q^Z}$ where $D_+(Q^Z) = R^+_{Q,\mathbb{C}}$, and $R^+_{Q^Z} \cup R_{Q^Z,\mathbb{R}}$ is a parabolic set of roots, and the corresponding parabolic $\mathfrak{q}$ is σ-stable.

By [Vog83, Proposition 2.1] (which in turn cites [Mat79]), for each K-conjugacy class of σ-stable Cartan subalgebras, there is a Cartan $\mathfrak{c}$ that is also σ'-stable. So we can choose $\mathfrak{c}$ to be so at the beginning. In fact, if I'm not mistaken, all real parabolic complexifies into a σ-split parabolic.

Combined with the Duality Theorem [Hec+87] and induction in stages, we obtain the following description. Let τ^L be a connection on Q^L compatible with $\lambda+\rho$, let $x \in Q^L$ so that $\mathfrak{b} = \mathfrak{b}_x \supseteq \mathfrak{c}$, and let $s = \dim(\mathfrak{k} \cap [\mathfrak{b}, \mathfrak{b}]) = \frac{1}{2}|R_{Q^L,CI}|$,

$$H^p(X, \mathcal{I}(Q^L, \tau^L))^\vee = R^{s-p} I^{(\mathfrak{g},K)}_{(\mathfrak{b},T)}\big((T_x\tau^L)^\vee \underset{\mathbb{C}}{\otimes} T_x\omega_X\big) \tag{1.12}$$

$$= H^{s-p} RI^{(\mathfrak{g},K)}_{(\mathfrak{p},K\cap P)} RI^{(\mathfrak{p},K\cap P)}_{(\mathfrak{b},T)}\big((T_x\tau^L)^\vee \underset{\mathbb{C}}{\otimes} T_x\omega_X\big). \tag{1.13}$$

(Here I should've written $\varphi^{-1}(P)$ instead of $K \cap P$ since K is not a subgroup of G, but $K \cap P$ is so much more intuitive and I'm reluctant to change this notation). Here RI denotes cohomological induction. In Part II of these talks we will recognize the first stage induction as the limit of discrete series, and the second stage induction as the usual parabolic induction. But first let's take care of what happens for non-Langlands orbits.

1.2. **Intertwining functor: translation to Langlands orbit.**

Intertwining functor relates standard modules on non-Langlands orbits to those on Langlands orbits.

Recall that diagonal G orbits on $X \times X$ are parameterized by W, where the orbit Z_w corresponding to w consists of pairs (x_1, x_2) such that $\mathfrak{b}_{x_2}$ is in relative position w with respect to $\mathfrak{b}_{x_1}$. This means that for any common Cartan $\mathfrak{c}$ of $\mathfrak{b}_{x_1}$ and $\mathfrak{b}_{x_2}$, the set of position roots R^+_2 of $(\mathfrak{g}, \mathfrak{c})$ defined by $\mathfrak{b}_{x_2}$ can be obtained by R^+_1 by $R^+_2 = wR^+_1$. Denote the projections by

$$X \xleftarrow{p_1} Z_w \xrightarrow{p_2} X. \tag{1.14}$$

The main tool is

Definition 1.15. Given $w \in W$, the **intertwining functor** is

$$LI_w : D^b(\mathcal{D}_\lambda) \longrightarrow D^b(\mathcal{D}_{w\lambda}), \quad \mathcal{F} \mapsto q_{1+}\Big(q_1^*\mathcal{O}_X(\rho - w\rho) \underset{\mathcal{O}_{Z_w}}{\otimes} q_2^+\mathcal{F}\Big) \tag{1.16}$$

where D^b is the category of all $\mathcal{D}_\lambda$-modules. It restricts to a functor between full subcategories of complexes with quasi-coherent cohomologies.

Theorem 1.17 ([Mil, 3.3.23]). *If λ is antidominant w.r.t. roots in Σ^+_w (defined in (1.25), then $R\Gamma \circ LI_w \cong R\Gamma$ on $\mathcal{D}_\lambda$-modules.*

Upshot 1.18. *For any orbit Q there is a Langlands orbit Q^L attached to the same Cartan and a certain choice of w such that $LI_w\mathcal{I}(Q, \tau) = \mathcal{I}(Q^L, \tau^L)$.*

In general, $\operatorname{Supp} LI_w\mathcal{F} \subseteq p_1(p_2^{-1}(\operatorname{Supp}\mathcal{F}))$, and the latter set has dimension less than or equal to $\dim \operatorname{Supp}\mathcal{F} + \ell(w)$. Also, strict containment can hold.

Digression on transversality. If $\dim p_1(p_2^{-1}(Z)) = \dim Z + \ell(w)$, we say w is **transversal to** Z.

Example 1.19. $G_0 = \mathbf{SL}(2,\mathbb{R})$, $(\mathfrak{g}, K) = (\mathfrak{sl}(2,\mathbb{C}), \mathbb{C}^*)$, $X = \mathbb{P}^1$. The K-orbits $\{0\}$, $\{\infty\}$ correspond to the compact Cartan, and $\mathbb{C}^* \subset X = \mathbb{P}^1$ corresponds to the split Cartan. So $\{0\}$ is a Langlands orbit. α is a non-compact imaginary root on the compact Cartan, and s_α is transversal to $\{0\}$. In this case $p_1(p_2^{-1}(\{0\})) = \mathbb{C}^* \cup \{\infty\}$ which contains an orbit attached to a different Cartan.

Example 1.20. $G_0 = \mathbf{SL}(2,\mathbb{C})$, $(\mathfrak{g}, K) = (\mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C}), \Delta\mathbf{SL}(2,\mathbb{C}))$, $X = \mathbb{P}^1 \times \mathbb{P}^1$. There is only one K-conjugacy class of Cartan: the one of $\mathfrak{c} = \mathfrak{c}_0 \times \mathfrak{c}_0 = \mathfrak{t} \oplus \mathfrak{a}$ where $\mathfrak{c}_0$ is a Cartan in $\mathfrak{g}_0$, $\mathfrak{t} = \Delta\mathfrak{c}_0$ and $\mathfrak{a} = \nabla\mathfrak{c}_0$ (skew diagonal). Both K-orbits $\Delta\mathbb{P}_1$ and $(\mathbb{P}^1 \times \mathbb{P}^1) - \Delta\mathbb{P}^1$ correspond to this Cartan. α is now a complex root on $\mathfrak{h}$ and s_α is transversal to $\Delta\mathbb{P}^1$ with $p_1(p_2^{-1}(\Delta\mathbb{P}^1)) = (\mathbb{P}^1 \times \mathbb{P}^1) - \Delta\mathbb{P}^1$.

In general, if s_α is simple and transversal to an orbit Q, then it is either Q-non-compact imaginary or Q-complex. We give a proof of this. The geometry discussed here will be useful later.

Let $\alpha \in \Sigma \subset \mathfrak{h}^*$ (note that now $\alpha \notin \mathfrak{c}^*$). Let $\mathfrak{c}$ be a σ-stable Cartan in $\mathfrak{g}$, Q a K-orbit attached to $\mathfrak{c}$, and $x \in Q$. Let $\Sigma_{Q,CI}, \ldots$ be the pullback of $R_{Q,CI}, \ldots$ along the specialization map $s_x : \mathfrak{h}^* \to \mathfrak{c}^*$ at x. $\Sigma_{Q,CI}, \ldots$ depends on Q be does not depend on $x \in Q$. In the same way we denote σ_Q be the involution on $\mathfrak{h}^*$ obtained as the pullback of σ along s_x. The set $D_+(Q)$ defined before also has an analogue in Σ^+ which we still call $D_+(Q)$. Namely

$$D_+(Q) = \{\alpha \in \Sigma^+_{Q,\mathbb{C}} \mid \sigma_Q\alpha \in \Sigma^+\}. \tag{1.21}$$

Consider the Cartesian diagram

$$\begin{tikzcd} Z_{s_\alpha} \cup \Delta X \arrow[r, "q_2"] \arrow[d, "q_1"] & X \arrow[d, "p_\alpha"] \\ X \arrow[r, "p_\alpha"] & X_\alpha \end{tikzcd} \tag{1.22}$$

where X_α is the partial flag variety corresponding to α. Write $O = p_\alpha(Q)$. Then $q_1(q_2^{-1}(Q)) = p_\alpha^{-1}(p_\alpha(Q)) = p_\alpha^{-1}(O)$. So s_α is transversal to Q iff. $\dim Q+1 = \dim p_1(p_2^{-1}(Q)) = \dim q_1(q_2^{-1}(Q)) = \dim p_\alpha^{-1}(O) = \dim O + 1$ iff. $\dim Q = \dim O$, i.e. Q is transversal to fibers of p_α. This condition can be checked on fibers of p_α: if $y = p_\alpha(x)$, then we can extend the above diagram to

$$\begin{tikzcd} & Q \arrow[d] & Q \cap p_\alpha^{-1}(y) \arrow[l] \arrow[d] \\ X \arrow[d, "p_\alpha"] & p_\alpha^{-1}(O) \arrow[l] \arrow[d] & p_\alpha^{-1}(y) \arrow[l] \arrow[d] \\ X_\alpha & O \arrow[l] & \{y\} \arrow[l] \end{tikzcd} . \tag{1.23}$$

Since $Q \to p_\alpha^{-1}(O) \to O$ is K-equivariant, $\dim Q = \dim O$ iff. $\dim Q \cap p_\alpha^{-1}(y) = 0$ iff. $Q \cap p_\alpha^{-1}(y)$ is a finite union of points. In general $Q \cap p_\alpha^{-1}(y)$ is a $S_y = \mathrm{Stab}_K(y)$-orbit in $p_\alpha^{-1}(y)$. So we need to study the S_y-orbit structure on $p_\alpha^{-1}(y)$.

Let $P_y = \mathrm{Stab}_G(y)$ and $B_x = \mathrm{Stab}_G(x)$. Then $p_\alpha^{-1}(y) \cong P_y/B_x \cong \mathbb{P}^1$ and can be viewed as the flag variety of a quotient $P_y/R_y \cong \mathbf{PSL}(2,\mathbb{C})$. The S_y-orbit structure is then determined by the image of S_y in $S_y \to P_y \to P_y/R_y \cong \mathbf{PSL}(2,\mathbb{C})$.

Lemma 1.24 ([Hec+, 6.5])**.**

(1) *If α is compact* Q*-imaginary, then the map* $S_y \to \mathbf{PSL}(2,\mathbb{C})$ *is surjective, and* S_y *acts transitively on* X_O.

(2) *If α is noncompact Q-imaginary, the image of $S_y \to \mathbf{PSL}(2,\mathbb{C})$ is the diagonal subgroup (a torus), and $Q \cap p_\alpha^{-1}(y)$ is one or two points.*
(3) *If α is Q-real, the image of $S_y \to \mathbf{PSL}(2,\mathbb{C})$ is the diagonal subgroup (a torus), and $Q \cap p_\alpha^{-1}(y)$ is isomorphic to $\mathbb{C}^*$.*
(4) *If α is Q-complex and $\alpha \in D_+(Q)$, the image of $S_y \to \mathbf{PSL}(2,\mathbb{C})$ has one dimensional unipotent radical whose Lie algebra is the image of $(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{\sigma\alpha})^\sigma$, and $Q \cap p_\alpha^{-1}(y)$ is a singleton.*
(5) *If α is Q-complex and $\alpha \notin D_+(Q)$, the image of $S_y \to \mathbf{PSL}(2,\mathbb{C})$ has one dimensional unipotent radical, and $Q \cap p_\alpha^{-1}(y)$ is isomorphic to $\mathbb{C}$.*

The proof is done by analyzing the Lie algebra map $\mathfrak{k} \cap \mathfrak{p}_y = \mathfrak{s}_y \to \mathfrak{sl}(2,\mathbb{C})$ by looking at root subspaces corresponding to different types of roots. The details are left as exercise.

As a result, if we want s_α to be transversal to Q, we need α to be noncompact Q-imaginary or Q-complex with $\alpha \in D_+(Q)$.

Back to our story. To achieve our goal we will find a $w = s_{\alpha_k} \cdots s_{\alpha_1}$ that is a product of complex simple reflections s_{α_i}'s such that

$$\alpha_1 \in D_+(Q),\ \alpha_2 \in D_+(Q_{s_{\alpha_1}}),\ \alpha_3 \in D_+((Q_{s_{\alpha_1}})_{s_{\alpha_2}}),\ \ldots$$

There are two slight issues in this process. The first issue is to calculate $D_+(Q_{s_{\alpha_1}})$ since s_{α_1} may send complex roots to other types of roots. The second issue is that there may not exist a simple root in $D_+(Q)$. The first task is left as an exercise.

Instead of finding simple reflections one at a time, we can actually find w directly.

$$\Sigma_w^+ = \{\alpha \in \Sigma^+ \mid w\alpha \notin \Sigma^+\} = \Sigma^+ \cap -w^{-1}\Sigma^+. \tag{1.25}$$

Lemma 1.26 ([Hec+, 5.4])**.** *For any set Σ_Q^+ of positive roots, there exists $w \in W$ such that $w^{-1}\Sigma_Q^+$ is a set of positive roots of Langlands type and*

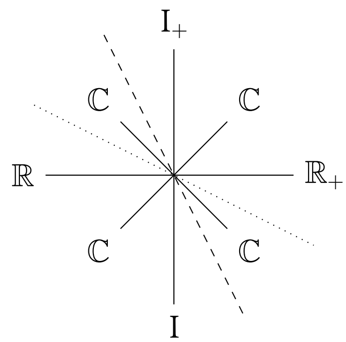
$$\Sigma_{Q,w}^+ \cap \sigma_Q(\Sigma_{Q,w}^+) = \varnothing, \quad \Sigma_{Q,w}^+ \cup \sigma_Q(\Sigma_{Q,w}^+) = D_+(Q). \tag{1.27}$$

Sketch of proof. First one chooses a set $\Sigma_{Q^L}^+$ of positive roots of Langlands type such that

$$\Sigma_Q^+ - \Sigma_{Q^L}^+ \subseteq D_+(Q), \tag{1.28}$$

i.e. for each pair of complex roots $\{\alpha, \sigma_Q\alpha\}$ in $D_+(Q)$, we include one of them in $\Sigma_{Q^L}^+$ and exclude the other one from $\Sigma_{Q^L}^+$. We give an imprecise but intuitive description on how this can be achieved. Write $\mathfrak{h}^*$ as $(\mathfrak{h}^*)|_{\sigma_Q=1} \oplus (\mathfrak{h}^*)|_{\sigma_Q=-1}$. The set Σ_Q^+ determines a positive direction in both $(\mathfrak{h}^*)|_{\sigma_Q=1}$ and $(\mathfrak{h}^*)|_{\sigma_Q=-1}$. If a pair of complex roots $\{\alpha, \sigma_Q\alpha\}$ is in $D_+(Q)$, then they are symmetrical w.r.t. $(\mathfrak{h}^*)|_{\sigma_Q=1}$. If we think of Σ_Q^+ and Σ_Q^- as separated by a hyperplane H in $\mathfrak{h}^*$, we can rotate H closer and closer to but does not overlap with $(\mathfrak{h}^*)|_{\sigma_Q=1}$ (i.e. in a way that makes the angle between $(\mathfrak{h}^*)|_{\sigma_Q=1}$ and H is getting smaller and the angle between $(\mathfrak{h}^*)|_{\sigma_Q=-1}$ and H is getting bigger). Eventually α and $\sigma_Q\alpha$ will be separated by the plane. The new set of positive roots determined by the rotated satisfies our requirement.

Look at the following example.



(1.29)

If Σ_Q^+ is determined by the dotted hyperplane, then we rotate it to the dashed hyperplane and obtain $\Sigma_{Q^L}^+$.

Let $w^{-1} \in W$ be such that $w^{-1}\Sigma_Q^+ = \Sigma_{Q^L}^+$. Then by construction, $\Sigma_{Q,w}^+ = \Sigma_Q^+ \cap -w^{-1}\Sigma_Q^+ = \Sigma_Q^+ \cap \Sigma_{Q^L}^- = \Sigma_Q^+ - \Sigma_{Q^L}^+$ which consists of exactly half of $D_+(Q)$, one from each pair $\{\alpha, \sigma_Q\alpha\}$. So $\sigma_Q(\Sigma_{Q,w}^+)$ consists of the other half of $D_+(Q)$, and (1.27) is clear. ∎

Note that such w is a product of complex simple reflections: if $w = w's_\alpha$ with $\ell(w) = \ell(w')+1$,

$$\{\alpha\} = \Sigma_{Q,s_\alpha}^+ \subseteq w^{-1}\Sigma_{Q,w'}^+ \cup \Sigma_{Q,s_\alpha}^+ = \Sigma_{Q,w}^+ \subseteq D_+(Q) \subset \Sigma_{Q,\mathbb{C}} \tag{1.30}$$

where the second equality follows from [Bou02, Ch.VI §1 no.6, Cor.2 of Prop.17].

Under this choice of w, we have desired transverality.

Proposition 1.31 ([Hec+, 6.9]). *Let Q be a K-orbit attached to $\mathfrak{c}$ and Σ_Q^+. Let $w \in W$ be such that $\Sigma_{Q,w}^+ \subseteq D_+(Q)$ and $\Sigma_{Q,w}^+ \cap \sigma_Q(\Sigma_{Q,w})^+ = \varnothing$. Let $X \xleftarrow{p_1} Z_w \xrightarrow{p_2} X$ be as before. Then*

(1) *$p_2^{-1}(Q)$ is a K-orbit in Z_w.*
(2) *$p_1 : p_2^{-1}(Q) \to Q_w := p_1(p_2^{-1}(Q))$ is an isomorphism of K-orbits.*
(3) *Q_w is attached to $\mathfrak{c}$ and $w^{-1}\Sigma_Q^+$.*

Proof. We prove this by induction on $\ell(w)$. Let $x \in Q$.

Consider the base case where $w = s_\alpha$ is a simple reflection. Since the map $p_2 : p_2^{-1}(Q) \to Q$ is K-equivariant, $p_2^{-1}(Q)$ is a single K-orbit iff. the fiber $p_2^{-1}(x)$ over x is a single S_x-orbit, where $S_x = \mathrm{Stab}_K(x)$. If this is the case, then $p_1 : p_2^{-1}(Q) \to p_1(p_2^{-1}(Q))$ is a K-equivariant map onto another K-orbit. This is an isomorphism iff. its restriction to a fiber of p_1 is bijective.

Extend the previous Cartesian diagrams to

$$\begin{tikzcd}
 & p_2^{-1}(x) \cup \{x\} \arrow[rr, "p_2 \cup \mathrm{Id}"] \arrow[dl] \arrow[dd, "p_1|_{p_2^{-1}(x)\cup\{x\}}"] & & \{x\} \arrow[dl] \arrow[dd] \\
Z_{s_\alpha} \cup \Delta X \arrow[rr, "q_2"] \arrow[dd, "q_1"] & & X \arrow[dd, "p_\alpha"] & \\
 & p_\alpha^{-1}(y) \arrow[rr] \arrow[dl] & & \{y\} \arrow[dl] \\
X \arrow[rr, "p_\alpha"] & & X_\alpha &
\end{tikzcd} \tag{1.32}$$

The squares on the front, top, bottom and back faces are Cartesian, and the vertical arrows on the back square are isomorphisms.

By our assumption, $\{\alpha\} = \Sigma^+_{Q,s_\alpha} \subseteq D_+(Q)$. By 1.24(4), the image of $S_y \to \mathbf{PSL}(2,\mathbb{C})$ has one-dimensional unipotent radical whose Lie algebra is the image of $(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{\sigma\alpha})^\sigma$, and its orbits on $p_\alpha^{-1}(y)$ are $\{y\}$ and $p_\alpha^{-1}(y) - \{y\}$. Since both α and $\sigma\alpha$ are positive, the Lie algebra of S_x also contains $(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{\sigma\alpha})^\sigma$. Hence the image of $S_x \hookrightarrow S_y \to \mathbf{PSL}(2,\mathbb{C})$ also has a one-dimensional unipotent radical. So the orbits of S_x and S_y on $p_\alpha^{-1}(y)$ are the same. By the S_x-equivariant isomorphism $p_2^{-1}(x) \cup \{x\} \xrightarrow{\sim} p_\alpha^{-1}(y)$, we see that $p_2^{-1}(x)$ is a single S_x-orbit. Thus $p_2^{-1}(Q)$ is a K-orbit.

Also, from the same arrow $p_2^{-1}(x) \cup \{x\} \xrightarrow{\sim} p_\alpha^{-1}(y)$ we see that $p_1|_{p_2^{-1}(Q)}$ is an isomorphism when restricted to a point $x \in p_\alpha^{-1}(y) - \{y\} \subset p_1(p_2^{-1}(Q))$. Hence $p_2^{-1}(Q) \to p_1(p_2^{-1}(Q))$ is an isomorphism. This proves (1) and (2) for $w = s_\alpha$.

It remains to show that $p_1(p_2^{-1}(Q))$ is attached to $\mathfrak{c}$. It is enough to show that there is a point $x' \in Q_\alpha$ so that $\mathfrak{b}_{x'} \supseteq \mathfrak{c}$. Define $\mathfrak{b}_{x'}$ to be the opposite Borel to $\mathfrak{b}_x \supseteq \mathfrak{c}$. Then the orbit containing x' is attached to $\mathfrak{c}$ and x' is in relative position s_α w.r.t. x. But the above argument shows that x' is a point in $p_1(p_2^{-1}(Q))$. Hence $p_1(p_2^{-1}(Q))$ is attached to $\mathfrak{c}$ and by construction of x' corresponds to $s_\alpha R^+_Q$

Now the inductive step. The detailed combinatorics will be omitted. We will only explain what is required to prove. Write $w = w's_\alpha$ with $\ell(w) = \ell(w') + 1$. By combinatorial properties of Σ^+_w, we know $D_+(Q) \supseteq \Sigma^+_w \supseteq \Sigma^+_{s_\alpha} = \{\alpha\}$. So s_α satisfies the conditions of the proposition with respect to Q. We also need w' to satisfy conditions of the proposition w.r.t. Q_{s_α}. More explicitly, we need

$$\Sigma^+_{Q_{s_\alpha},w'} \subseteq D_+(Q_{s_\alpha}) \text{ and } \Sigma^+_{Q_{s_\alpha},w'} \cap \sigma_{Q_{s_\alpha}}(\Sigma^+_{Q_{s_\alpha},w'}) = \varnothing.$$

These can be proven by using the condition $\Sigma^+_{Q,w} \cap \sigma_Q(\Sigma^+_{Q,w}) = \varnothing$ (which we did not use in the base case) and doing combinatorial work.

Then, by inductive assumption, we have the following diagram with solid arrows

$$\begin{tikzcd}
 & & & Z_w \arrow[dl, dotted] \arrow[dr, dotted] & & \\
 & & Z_{w'} \arrow[dl] \arrow[dr] & & Z_{s_\alpha} \arrow[dl] \arrow[dr] & \\
 & X & & X & & X \\
 & & \widetilde{Q_{s_\alpha}} \times_{Q_{s_\alpha}} \tilde{Q} \arrow[uuur, dotted] \arrow[dl, dotted] \arrow[dr, dotted] & & & \\
 & \widetilde{Q_{s_\alpha}} \arrow[uuur] \arrow[dl, "\cong"'] \arrow[dr] & & \tilde{Q} \arrow[uuur] \arrow[dl, "\cong"'] \arrow[dr] & & \\
 (Q_{s_\alpha})_{w'} \arrow[uuur] & & Q_{s_\alpha} \arrow[uuur] & & Q \arrow[uuur] &
\end{tikzcd} \tag{1.33}$$

To complete the proof, notice that we can fill in the dotted maps so that all squares facing in the front or up-right direction are Cartesian. Moreover, the two outside maps from Z_w to X are the usual ones. Thus $\widetilde{Q_{s_\alpha}} \times_{Q_{s_\alpha}} \tilde{Q}$ is a subset in Z_w which is the preimage of Q under the second projection $Z_w \to X$, and the map $\widetilde{Q_{s_\alpha}} \times_{Q_{s_\alpha}} \tilde{Q} \to (Q_{s_\alpha})_{w'}$ is an isomorphism. This completes the proof. ∎

On the level of $\mathcal{D}$-modules, by a usual base change argument, it is easy to show the following.

Corollary 1.34. *Under the assumption of the proposition,*

(1) *There is a bijection $\tau \mapsto \tau_w$ between irreducible K-homogeneous connections on Q compatible with $\lambda + \rho$ and those on Q_w compatible with $w\lambda + \rho$.*
(2) $LI_w \mathcal{I}(Q, \tau) = \mathcal{I}(Q_w, \tau_w)$.

Proof. Consider the diagram

$$\begin{tikzcd} Q_w \arrow[d, "i_{Q_w}"'] & p_2^{-1}(Q) \arrow[l, "\cong"'] \arrow[r, "q_w"] \arrow[d, "j"] & Q \arrow[d, "i_Q"] \\ X & Z_w \arrow[l, "p_1"'] \arrow[r, "p_2"] & X \end{tikzcd} \tag{1.35}$$

where the square on the right is Cartesian. Identify Q_w with $p_2^{-1}(Q)$ via the isomorphism. By base change

$$LI_w \mathcal{I}(Q, \tau) = \mathcal{O}_X(\rho - w\rho) \underset{\mathcal{O}_X}{\otimes} p_{1+} p_2^{+} i_{Q+} \tau \tag{1.36}$$

$$= \mathcal{O}_X(\rho - w\rho) \underset{\mathcal{O}_X}{\otimes} p_{1+} p_2^{!} i_{Q+} \tau[m] \tag{1.37}$$

$$= \mathcal{O}_X(\rho - w\rho) \underset{\mathcal{O}_X}{\otimes} p_{1+} j_{+} q_w^{!} \tau[m] \tag{1.38}$$

$$= \mathcal{O}_X(\rho - w\rho) \underset{\mathcal{O}_X}{\otimes} i_{Q_w+} q_w^{+} \tau \tag{1.39}$$

$$= i_{Q_w+}\big(i_{Q_w}^{*} \mathcal{O}_X(\rho - w\rho) \underset{\mathcal{O}_{Q_w}}{\otimes} q_w^{+} \tau\big) \tag{1.40}$$

$$= \mathcal{I}(Q_w, \tau_w) \tag{1.41}$$

where $\tau_w := i_{Q_w+}^{*} \mathcal{O}_X(\rho - w\rho) \otimes_{\mathcal{O}_{Q_w}} q_w^{+} \tau$. It remains to show that $\tau \mapsto \tau_w$ is a bijection between irreducible connections. By taking fiber over points $x \in Q$, $x' \in Q_w$ in the orbits, this map corresponds to the map

$$\mathrm{Irrep}(S_x) \to \mathrm{Irrep}(S_{x'}) \xrightarrow{\text{twist by a character}} \mathrm{Irrep}(S_{x'}) \tag{1.42}$$

where the first map is the restriction along $K \cap B_{x'} = S_{x'} \subset S_x = K \cap B_x$. Both S_x and $S_{x'}$ are solvable, so their irreducible representations are determined by characters on maximal torus. Since both Q and Q_w are attached to $\mathfrak{c} = \mathfrak{t} \oplus \mathfrak{a}$, (some conjugate of) T (the subgroup of K corresponding to $\mathfrak{t}$) is a common maximal torus of S_x and $S_{x'}$. So (1.42) is a bijection. This completes the proof. ∎

Combined with 1.26, we see that

Corollary 1.43. *For every K-orbit Q, there exists a $w \in W$ transversal to Q such that*

- *$Q_w = Q^L$ is a Langlands orbit, and*
- *$LI_w \mathcal{I}(Q, \tau) = \mathcal{I}(Q^L, \tau^L)$, where τ^L is some K-homogeneous connection on Q^L.*

More precisely, w can be chosen such that $w^{-1}\Sigma_Q^+$ is a set of positive roots of Langlands type,

$$\Sigma_{Q,w}^+ \cap \sigma_Q(\Sigma_{Q,w}^+) = \varnothing, \quad \Sigma_{Q,w}^+ \cup \sigma_Q(\Sigma_{Q,w}^+) = D_+(Q) \tag{1.27}$$

holds, Q^L is determined by $w^{-1}\Sigma_Q^+$, and

$$\tau^L = \tau_w = i_{Q_w+}^{*} \mathcal{O}_X(\rho - w\rho) \underset{\mathcal{O}_{Q_w}}{\otimes} q_w^{+} \tau \tag{1.44}$$

as in the proof of 1.34.

If λ is antidominant regular to start with, then

$$\begin{aligned} H^0(X,\mathcal{I}(Q,\tau))^\vee &= \left(H^0 R\Gamma \mathcal{I}(Q,\tau)\right)^\vee & (1.45)\\ &\cong \left(H^0 R\Gamma\, LI_w \mathcal{I}(Q,\tau)\right)^\vee & (1.46)\\ &= H^0(X,\mathcal{I}(Q^L,\tau^L))^\vee & (1.47)\\ &= H^s RI^{(\mathfrak{g},K)}_{(\mathfrak{p},K\cap P)} RI^{(\mathfrak{p},K\cap P)}_{(\mathfrak{b},T)}\left((T_x\tau^L)^\vee \underset{\mathbb{C}}{\otimes} T_x\omega_X\right) & (1.48)\end{aligned}$$

where $s = \frac{1}{2}|R_{Q^L,CI}|$, $\mathfrak{b} \supseteq \mathfrak{c}$ corresponds to a point in Q^L, and $\mathfrak{p}$ is the σ-split parabolic corresponding to $R_{Q^L,I} \cup R^+_{Q^L}$.

2. Induction in stages

Recall that $x \in Q^L$ and $\mathfrak{b}$ is the corresponding Borel subalgebra which we require to contain the σ-stable Cartan subalgebra $\mathfrak{c}$. Write $E = (T_x\tau^L)^\vee \otimes_{\mathbb{C}} T_x\omega_X$ which is a $(\mathfrak{b},T)$-module on which $\mathfrak{n}$ acts trivially.

The module

$$H^s RI^{(\mathfrak{g},K)}_{(\mathfrak{p},K\cap P)} RI^{(\mathfrak{p},K\cap P)}_{(\mathfrak{b},T)} E \tag{2.1}$$

can be computed by the Grothendieck spectral sequence. As it will turn out, $RI^{(\mathfrak{p},K\cap P)}_{(\mathfrak{b},T)} E$ is in fact concentrated in degree s and is a limit of discrete series representation of a Levi factor of P, and $I^{(\mathfrak{g},K)}_{(\mathfrak{p},K\cap P)}$ is the same as parabolic induction and is exact. Hence the spectral sequence collapses.

2.1. **First stage: limit of discrete series of the Levi.** In this section we will show that $RI^{(\mathfrak{p},K\cap P)}_{(\mathfrak{b},T)} E$ is concentrated in degree s and is a limit of discrete series representation.

Let

$$\begin{aligned} \mathfrak{p} &= \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{u} & (2.2)\\ &= ([\bar{\mathfrak{d}},\bar{\mathfrak{d}}] \oplus \mathfrak{d}) \oplus \mathfrak{a} \oplus \mathfrak{u} & (2.3)\\ &= ([\bar{\mathfrak{d}},\bar{\mathfrak{d}}] \oplus \mathfrak{t} \oplus [\mathfrak{d},\mathfrak{d}]) \oplus \mathfrak{a} \oplus \mathfrak{u} & (2.4)\end{aligned}$$

where

- $\mathfrak{c} = \mathfrak{t} \oplus \mathfrak{a}$ is the decomposition into σ-eigenspaces,
- $\mathfrak{p} = (\mathfrak{m} \oplus \mathfrak{a}) \oplus \mathfrak{u}$ is the $\mathfrak{c}$-stable Levi decomposition,
- $\mathfrak{m}$ is the semisimple part of the Levi $\mathfrak{m} \oplus \mathfrak{a}$ of $\mathfrak{p}$, spanned by $\mathfrak{t}$ and imaginary root spaces.
- ($\mathfrak{p} \cap \mathfrak{k}$ is spanned by $\mathfrak{t}$ and compact imaginary root spaces. So $\mathfrak{p} \cap \mathfrak{k} \subseteq \mathfrak{m}$).
- $\mathfrak{d} = \mathfrak{m} \cap \mathfrak{b}$ is a Borel subalgebra of $\mathfrak{m}$.

Our first goal is to insert our Levi into the game.

Lemma 2.5. *For any* $(\mathfrak{c},T)$*-module* E*,*

$$RI^{(\mathfrak{p},P\cap K)}_{(\mathfrak{b},T)} E \cong RI^{(\mathfrak{m}\oplus\mathfrak{a},P\cap K)}_{(\mathfrak{d}\oplus\mathfrak{a},T)} E \tag{2.6}$$

in $D^b(\mathfrak{p}, P \cap K)$.

Proof. Consider the following diagram, where $\mathrm{For}^{\amalg}_{\%}$ denotes the forgetful functor from the category of $\amalg$-modules to the category of %-modules, and $(-)_{\mathfrak{n}}$ denotes the derived functor of taking

$\mathfrak{n}$-coinvariants. Similarly for $(-)_{\mathfrak{u}}$.

$$\begin{tikzcd}
D^b(\mathfrak{b},T) \arrow[rr, shift left, "\mathrm{RI}^{(\mathfrak{p},P\cap K)}_{(\mathfrak{b},T)}"'] \arrow[rr, phantom, shift left=3, "\perp"] \arrow[d, dashed, shift right, "(-)_{\mathfrak{n}}"'] \arrow[d, phantom, "\dashv"] & & D^b(\mathfrak{p},P\cap K) \arrow[ll, dashed, shift left=5, "\mathrm{For}^{(\mathfrak{p},P\cap K)}_{(\mathfrak{b},T)}"'] \arrow[d, dashed, shift right, "(-)_{\mathfrak{u}}"'] \arrow[d, phantom, "\dashv"] \\
D^b(\mathfrak{c},T) \arrow[u, shift right, "\mathrm{For}^{(\mathfrak{b},T)}_{(\mathfrak{c},T)}"'] \arrow[r, shift right, "\mathrm{For}^{(\mathfrak{d}\oplus\mathfrak{a},T)}_{(\mathfrak{c},T)}"'] \arrow[r, phantom, "\perp"] & D^b(\mathfrak{d}\oplus\mathfrak{a},T) \arrow[l, dashed, shift right, "(-)_{[\mathfrak{d},\mathfrak{d}]}"'] \arrow[r, shift right, "\mathrm{RI}^{(\mathfrak{m}\oplus\mathfrak{a},P\cap K)}_{(\mathfrak{d}\oplus\mathfrak{a},T)}"'] \arrow[r, phantom, "\perp"] & D^b(\mathfrak{m}\oplus\mathfrak{a},P\cap K) \arrow[l, dashed, shift right, "\mathrm{For}^{(\mathfrak{m}\oplus\mathfrak{a},P\cap K)}_{(\mathfrak{d}\oplus\mathfrak{a},T)}"'] \arrow[u, shift right, "\mathrm{For}^{(\mathfrak{p},P\cap K)}_{(\mathfrak{m}\oplus\mathfrak{a},P\cap K)}"']
\end{tikzcd} \tag{2.7}$$

The module $\mathrm{RI}^{(\mathfrak{p},P\cap K)}_{(\mathfrak{b},T)}\mathrm{For}^{(\mathfrak{b},T)}_{(\mathfrak{c},T)}\tilde{E}_{-s_{\mathfrak{b}}(w\lambda-\rho)}$ is the image of $\mathbb{C}_{-s_{\mathfrak{b}}(w\lambda)} \in \mathrm{Mod}(\mathfrak{c},T)$ along the top solid path. Since $[\mathfrak{d},\mathfrak{d}] \oplus \mathfrak{u} = \mathfrak{n}$, $(-)_{[\mathfrak{d},\mathfrak{d}]} \circ (-)_{\mathfrak{u}} = (-)_{\mathfrak{n}}$. Hence the dotted arrows form a commutative diagram. The two paths formed by solid arrows are right adjoints to the two paths formed by dotted arrows, respectively. Hence the solid arrows also form a commutative diagram, from which the desired identity follows. ∎

Lemma 2.8. *Let E be a $(\mathfrak{d}\oplus\mathfrak{a},T)$-module on which $\mathfrak{a}$ acts by $\mu \in \mathfrak{a}^*$. Then*

$$\mathrm{RI}^{(\mathfrak{m}\oplus\mathfrak{a},P\cap K)}_{(\mathfrak{d}\oplus\mathfrak{a},T)}E \cong \left(\mathrm{RI}^{(\mathfrak{m},P\cap K)}_{(\mathfrak{d},T)}(E|_{(\mathfrak{d},T)})\right) \boxtimes \mathbb{C}_{\mu} \tag{2.9}$$

where $\mathbb{C}_{\mu}$ is considered as an $\mathfrak{a}$-module.

Proof. We first show the non-derived version, i.e. that the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Mod}(\mathfrak{d}\oplus\mathfrak{a},T) \arrow[r, "\mathrm{I}^{(\mathfrak{m}\oplus\mathfrak{a},P\cap K)}_{(\mathfrak{d}\oplus\mathfrak{a},T)}"] & \mathrm{Mod}(\mathfrak{m}\oplus\mathfrak{a},P\cap K) \\
\mathrm{Mod}(\mathfrak{d},T) \arrow[u, "-\boxtimes\mathbb{C}_{\mu}"] \arrow[r, "\mathrm{I}^{(\mathfrak{m},P\cap K)}_{(\mathfrak{d},T)}"] & \mathrm{Mod}(\mathfrak{m},P\cap K) \arrow[u, "-\boxtimes\mathbb{C}_{\mu}"']
\end{tikzcd} \tag{2.10}$$

By exactness of $-\boxtimes\mathbb{C}_{\mu}$ and composition of derived functors, we immediately get the commutative diagram.

$$\begin{tikzcd}
D^b(\mathfrak{d}\oplus\mathfrak{a},T) \arrow[r, "\mathrm{RI}^{(\mathfrak{m}\oplus\mathfrak{a},P\cap K)}_{(\mathfrak{d}\oplus\mathfrak{a},T)}"] & D^b(\mathfrak{m}\oplus\mathfrak{a},P\cap K) \\
D^b(\mathfrak{d},T) \arrow[u, "-\boxtimes\mathbb{C}_{\mu}"] \arrow[r, "\mathrm{RI}^{(\mathfrak{m},P\cap K)}_{(\mathfrak{d},T)}"] & D^b(\mathfrak{m},P\cap K) \arrow[u, "-\boxtimes\mathbb{C}_{\mu}"']
\end{tikzcd} \tag{2.11}$$

Since $\mathfrak{a} \circlearrowright E$ by μ, $E \cong (\mathrm{For}^{(\mathfrak{d}\oplus\mathfrak{a},T)}_{(\mathfrak{d},T)} E) \boxtimes \mathbb{C}_{\mu}$ as $(\mathfrak{d}\oplus\mathfrak{a},T)$-modules. (2.11) applied to $\mathrm{For}^{(\mathfrak{d}\oplus\mathfrak{a},T)}_{(\mathfrak{d},T)} E$ then produces the desired identity.

Write $U = \mathrm{For}^{(\mathfrak{d}\oplus\mathfrak{a},T)}_{(\mathfrak{d},T)}$. To show that the diagram commutes, we use Yoneda lemma, i.e. we show that there is an isomorphism

$$\mathrm{Hom}_{(\mathfrak{m}\oplus\mathfrak{a},P\cap K)}(W, \mathrm{I}^{(\mathfrak{m}\oplus\mathfrak{a},P\cap K)}_{(\mathfrak{d}\oplus\mathfrak{a},T)}(U\boxtimes\mathbb{C}_{\mu})) \cong \mathrm{Hom}_{(\mathfrak{m}\oplus\mathfrak{a},P\cap K)}(W, (\mathrm{I}^{(\mathfrak{m},P\cap K)}_{(\mathfrak{d},T)}U)\boxtimes\mathbb{C}_{\mu}) \tag{2.12}$$

for any $W \in \mathrm{Mod}(\mathfrak{m}\oplus\mathfrak{a},P\cap K)$.

Consider the set on the right side. Since $\mathfrak{a}$ is central in the pair $(\mathfrak{m}\oplus\mathfrak{a},P\cap K)$, we can define the μ-coinvariant of W as $W_{\mu} = W/(a-\mu(a))w \mid a \in \mathfrak{a}, w \in W\rangle$ which is isomorphic to $W_{\mu}|_{(\mathfrak{m},P\cap K)} \boxtimes \mathbb{C}_{\mu}$. Also because $\mathfrak{a}$ is central, any map in the Hom set on the right must factor through W_{μ}. Hence

$$\mathrm{Hom}_{(\mathfrak{m}\oplus\mathfrak{a},P\cap K)}(W, (\mathrm{I}^{(\mathfrak{m},P\cap K)}_{(\mathfrak{d},T)}U)\boxtimes\mathbb{C}_{\mu}) \tag{2.13}$$

$$= \mathrm{Hom}_{(\mathfrak{m}\oplus\mathfrak{a},P\cap K)}(W_\mu, (I^{(\mathfrak{m},P\cap K)}_{(\mathfrak{d},T)}U) \boxtimes \mathbb{C}_\mu) \tag{2.14}$$

$$= \mathrm{Hom}_{(\mathfrak{m}\oplus\mathfrak{a},P\cap K)}(W_\mu|_{(\mathfrak{m},P\cap K)} \boxtimes \mathbb{C}_\mu, (I^{(\mathfrak{m},P\cap K)}_{(\mathfrak{d},T)}U) \boxtimes \mathbb{C}_\mu) \tag{2.15}$$

$$= \mathrm{Hom}_{(\mathfrak{m},P\cap K)}(W_\mu|_{(\mathfrak{m},P\cap K)}, (I^{(\mathfrak{m},P\cap K)}_{(\mathfrak{d},T)}U)) \tag{2.16}$$

which, by adjunction, is equal to $\mathrm{Hom}_{(\mathfrak{d},T)}(W_\mu|_{(\mathfrak{d},T)}, U)$.

Now consider the Hom set on the left. By adjunction it is equal to $\mathrm{Hom}_{(\mathfrak{d}\oplus\mathfrak{a},T)}(W, U \boxtimes \mathbb{C}_\mu)$. Again since $\mathfrak{a}$ is central in $(\mathfrak{d} \oplus \mathfrak{a}, T)$, any map in this set factors through $W_\mu|_{(\mathfrak{d},T)} \boxtimes \mathbb{C}_\mu$. Hence this Hom set is isomorphic to $\mathrm{Hom}_{(\mathfrak{d},T)}(W_\mu|_{(\mathfrak{d},T)}, U)$ as well. This completes the proof. ∎

As a result

$$RI^{(\mathfrak{p},P\cap K)}_{(\mathfrak{b},T)}E \cong RI^{(\mathfrak{m}\oplus\mathfrak{a},P\cap K)}_{(\mathfrak{d}\oplus\mathfrak{a},T)}E \cong \left(RI^{(\mathfrak{m},P\cap K)}_{(\mathfrak{d},T)}(E|_{(\mathfrak{d},T)})\right) \boxtimes \mathbb{C}_\mu. \tag{2.17}$$

We want to identify the first factor on the right side with a limit of discrete series of $\mathfrak{m}$. We will use the geometric criterion for limit of discrete series.

Theorem 2.18 ([Hec+, 12.5, 12.6])**.** *Suppose* $\operatorname{rank}\mathfrak{g} = \operatorname{rank} K$. *Let* λ *be strongly antidominant (i.e.* $\operatorname{Re}\alpha^\vee(\lambda) \leqslant 0$ *for simple* α*'s),* Q *a closed* K*-orbit, and* τ *an irreducible* K*-homogeneous connection on* Q *compatible with* $\lambda+\rho$*. Then*

- *If* λ *is regular, then* $\Gamma(X, \mathcal{I}(Q,\tau))$ *is a discrete series representation.*
- *If* λ *is singular and* $\Gamma(X, \mathcal{I}(Q,\tau)) \neq 0$*, then* $\Gamma(X, \mathcal{I}(Q,\tau))$ *is a limit of discrete series representation.*

In our situation, suppose $\lambda \in \mathfrak{h}^*$ is strongly antidominant. Let

- $X_\mathfrak{m}$ be the flag variety of $\mathfrak{m}$,
- $\sigma_\mathfrak{m} = \sigma|_\mathfrak{m}$.

Then $\mathfrak{p}\cap\mathfrak{k}$ is the fixed points of $\sigma_\mathfrak{m}$, and $\mathfrak{t}$ is a common maximal torus in $\mathfrak{p}\cap\mathfrak{k}$ and $\mathfrak{m}$, so $\operatorname{rank}\mathfrak{m} = \operatorname{rank} P\cap K$.

- $Q_\mathfrak{m}$ be the $P\cap K$-orbit in $X_\mathfrak{m}$ containing a point z corresponding to the Cartan $\mathfrak{t}$ and the Borel $\mathfrak{d}$.

Since $\mathfrak{d}\cap\mathfrak{k}$ is spanned by $\mathfrak{t}$ and positive compact imaginary root spaces, which is a Borel subalgebra in K. Hence $Q_\mathfrak{m}$ is a closed orbit. Let

- $\rho_\mathfrak{d} \in \mathfrak{t}^*$ be the half sum of roots in $R^+ \cap R_I$,
- $\tau_\mathfrak{m}$ be a $P\cap K$-homogeneous connection on $Q_\mathfrak{m}$ determined by $T \circlearrowright E^\vee \otimes_\mathbb{C} T_z\omega_{X_\mathfrak{m}}$.

Recall that $E = (T_x\tau^L)^\vee \otimes_\mathbb{C} T_x\omega_X$. Here $\mathfrak{c}$ acts on $T_x\tau^L$ by specialization $s_x(w\lambda+\rho)$ of $w\lambda+\rho$ to x (because τ^L is compatible with $w\lambda+\rho$) and on $T_x\omega_X$ by the specialization $s_x(2\rho)$ of 2ρ. So $\mathfrak{c}$ acts on E by $-s_x(w\lambda+\rho)+s_x(2\rho) = -s_x(w\lambda-\rho)$. Therefore $\mathfrak{t}$ acts on $E^\vee \otimes_\mathbb{C} T_z\omega_{X_\mathfrak{m}}$ by $s_x(w\lambda-\rho)|_\mathfrak{t} + 2\rho_\mathfrak{d}$. Hence $\mathcal{I}(Q_\mathfrak{m}, \tau_\mathfrak{m})$ is a $\mathcal{D}_{s_x(w\lambda-\rho)|_\mathfrak{t}+\rho_\mathfrak{d}}$-module on $X_\mathfrak{m}$.

By duality theorem again,

$$H^q(X_\mathfrak{m}, \mathcal{I}(Q_\mathfrak{m},\tau_\mathfrak{m}))^\vee = R^{s'-q}I^{(\mathfrak{m},P\cap K)}_{(\mathfrak{d},T)}\left((E^\vee \underset{\mathbb{C}}{\otimes} T_z\omega_{X_\mathfrak{m}})^\vee \underset{\mathbb{C}}{\otimes} T_z\omega_{X_\mathfrak{m}}\right) = R^{s'-q}I^{(\mathfrak{m},P\cap K)}_{(\mathfrak{d},T)}E. \tag{2.19}$$

Here $s' = \dim(\mathfrak{p}\cap\mathfrak{k}) \cap [\mathfrak{d},\mathfrak{d}] = \frac{1}{2}|R_{CI}| = \dim \mathfrak{k}\cap[\mathfrak{b},\mathfrak{b}] = s$. So

$$H^q(X_\mathfrak{m}, \mathcal{I}(Q_\mathfrak{m},\tau_\mathfrak{m}))^\vee = R^{s-q}I^{(\mathfrak{m},P\cap K)}_{(\mathfrak{d},T)}E. \tag{2.20}$$

To use the criterion for limit of discrete series, we still need

Lemma 2.21. *Assume that* $\lambda \in \mathfrak{h}^*$ *is strongly antidominant. Then* $s_x(w\lambda-\rho)|_\mathfrak{t} + \rho_\mathfrak{d}$ *is strongly antidominant for the set of positive roots of* $(\mathfrak{m},\mathfrak{t})$ *determined by* $\mathfrak{d}$*.*

Proof. Let $x_Q \in Q$ denote a point which corresponds to a Borel subalgebra $\mathfrak{b}_Q$ containing $\mathfrak{c}$. Let $R_Q^+, R_{Q^L}^+ \subset R = R(\mathfrak{g}, \mathfrak{c})$ be the sets of positive roots determined by $\mathfrak{b}_Q$ and $\mathfrak{b} = \mathfrak{b}_x$, respectively. We identify the root system $R(\mathfrak{m}, \mathfrak{t})$ with R_I.

Then $s_x = s_{x_Q} \circ w^{-1}$ and

$$s_x(w\lambda - \rho)|_\mathfrak{t} + \rho_\mathfrak{d} = s_{x_Q}(w^{-1}(w\lambda - \rho))|_\mathfrak{t} + \rho_\mathfrak{d} = s_{x_Q}(\lambda)|_\mathfrak{t} - (w^{-1}s_{x_Q}(\rho))|_\mathfrak{t} + \rho_\mathfrak{d}. \tag{2.22}$$

We omit the $|_\mathfrak{t}$ from now. We want to show that for any $\beta \in R_{Q^L,I}^+$,

$$\operatorname{Re} \beta^\vee(s_x(w\lambda - \rho)|_\mathfrak{t} + \rho_\mathfrak{d}) = \operatorname{Re} \beta^\vee s_{x_Q}(\lambda) + \operatorname{Re} \beta^\vee(-w^{-1}s_{x_Q}(\rho) + \rho_\mathfrak{d}) \leqslant 0. \tag{2.23}$$

The term $-w^{-1}s_{x_Q}(\rho) + \rho_\mathfrak{d}$ is the half sum of roots in $w^{-1}R_Q^- \cup R_{Q^L,I}^+$. recall that by construction $R_{Q^L}^+ = w^{-1}R_Q^+$. So $-w^{-1}s_{x_Q}(\rho) + \rho_\mathfrak{d} = \sum(R_{Q^L}^- \cup R_{Q^L,I}^+) = \sum(R_{Q^L}^- - R_I)$ which is a sum of roots negative w.r.t. $\mathfrak{b}$ or $\mathfrak{d}$. Hence $\beta^\vee(-w^{-1}s_{x_Q}(\rho) + \rho_\mathfrak{d})$ is real and is $\leqslant 0$.

It remains to show that $\operatorname{Re} \beta^\vee s_{x_Q}(\lambda) \leqslant 0$. Since λ is strongly antidominant, $s_{x_Q}(\lambda)$ is strongly antidominant w.r.t. R_Q^+. Therefore it suffices to show that all the β's are in R_Q^+, i.e. that $R_{Q^L,I}^+ \subseteq R_Q^+$. To see this, write

$$R_{Q^L,I}^+ = R_{Q^L}^+ \cap R_I = w^{-1}R_Q^+ \cap R_I \tag{2.24}$$

and let β be in this set. Then $w\beta \in R_Q^+$. Hence $\beta = w^{-1}(w\beta) \in R_Q^+$ if and only if $w\beta \notin R_{Q,w^{-1}}^+ = -wR_{Q,w}^+$, which happens if and only if $-\beta \notin R_{Q,w}^+$. By our choice of w, $R_{Q,w}^+ \subseteq D_+(Q) \subseteq R_\mathbb{C}$. On the other hand, $\beta \in R_I$, so $-\beta$ is imaginary and cannot lie in $R_{Q,w}^+$. As a result $R_{Q^L,I}^+ \subseteq R_Q^+$ and $\operatorname{Re} \beta^\vee s_{x_Q}(\lambda) \leqslant 0$ for all $\beta \in R_{Q^L,I}^+$. This completes the proof. ∎

Assuming this, by 2.18, we see that if $\lambda \in \mathfrak{h}^*$ is strongly antidominant

$$\left(R^s I_{(\mathfrak{d},T)}^{(\mathfrak{m},P\cap K)} E\right)^\vee \tag{2.25}$$

is a (limit of) discrete series of $\mathfrak{m}$ whenever it is nonzero, and $R^{s-q} I_{(\mathfrak{d},T)}^{(\mathfrak{m},P\cap K)} E = 0$ for $q \neq 0$. Moreover, Harish-Chandra duality sends a limit of discrete series to another. As a result, we obtain

Proposition 2.26. *Let $\lambda \in \mathfrak{h}^*$ be strongly antidominant, Then*

$$RI_{(\mathfrak{b},T)}^{(\mathfrak{p},P\cap K)} E \cong \left(RI_{(\mathfrak{d},T)}^{(\mathfrak{m},P\cap K)} E|_{(\mathfrak{d},T)}\right) \boxtimes E|_\mathfrak{a} \tag{2.27}$$

is concentrated in degree $s = \frac{1}{2}|R_{CI}|$ and the first component is a limit of discrete series of $\mathfrak{m}$ whenever it is nonzero.

In particular the Grothendieck spectral sequence for (1.48) collapses at E_2 page:

$$H^0(X, \mathcal{I}(Q,\tau))^\vee = H^s RI_{(\mathfrak{p},K\cap P)}^{(\mathfrak{g},K)} RI_{(\mathfrak{b},T)}^{(\mathfrak{p},K\cap P)} \left((T_x\tau^L)^\vee \underset{\mathbb{C}}{\otimes} T_x\omega_X\right) \tag{2.28}$$

$$= I_{(\mathfrak{p},K\cap P)}^{(\mathfrak{g},K)} R^s I_{(\mathfrak{b},T)}^{(\mathfrak{p},K\cap P)} \left((T_x\tau^L)^\vee \underset{\mathbb{C}}{\otimes} T_x\omega_X\right). \tag{2.29}$$

2.2. **Second stage: parabolic induction.** Recall that our parabolic $\mathfrak{p}$ is the complexification of a parabolic $\mathfrak{p}_0$ in $\mathfrak{g}_0$. Let P_0 denote the corresponding group in G_0.

Let V denote the representation of M_0A_0 whose space of $P_0 \cap K_0$-finite vectors equals $RI_{(\mathfrak{d},T)}^{(\mathfrak{m},P\cap K)} E|_{(\mathfrak{d},T)} \boxtimes E|_\mathfrak{a}$, i.e.

$$V_{[P_0\cap K_0]} = RI_{(\mathfrak{d},T)}^{(\mathfrak{m},P\cap K)} E|_{(\mathfrak{d},T)} \boxtimes E|_\mathfrak{a}. \tag{2.30}$$

Let $\operatorname{Ind}_{P_0}^{G_0} V$ denote the classical (L^2, smooth or continuous) parabolic induction of V, i.e. the space of (L^2, smooth or continuous) functions $f : G_0 \to V$ so that $f(pg) = p \cdot f(g)$ for any $p \in P_0$,

$g \in G_0$. The G_0-action is given by $g \cdot f(-) = f(-g)$. The corresponding $(\mathfrak{g}, K)$-module is then $\left(\mathrm{Ind}_{P_0}^{G_0} V\right)_{[K]}$. There is an obvious map of $(\mathfrak{p}, P \cap K)$-modules

$$\left(\mathrm{Ind}_{P_0}^{G_0} V\right)_{[K]} \longrightarrow V, \quad f \mapsto f(1) \tag{2.31}$$

For a K-finite element f on the left side, it is in particular $P \cap K$-finite; since evaluation at $1 \in G_0$ sends finite dimensional subspaces of $\mathrm{Ind}_{P_0}^{G_0} V$ to finite dimensional ones, $f(1)$ is $P \cap K$-finite. Hence this map lands into $V_{[P\cap K]}$. By adjunction of cohomological induction, we obtain a map of $(\mathfrak{g}, K)$-modules

$$\Psi : \left(\mathrm{Ind}_{P_0}^{G_0} V\right)_{[K]} \longrightarrow I_{(\mathfrak{p}, P\cap K)}^{(\mathfrak{g},K)} V_{[P\cap K]}. \tag{2.32}$$

Explicitly, if we view $I_{(\mathfrak{p}, P\cap K)}^{(\mathfrak{g},K)} V_{[P\cap K]}$ as a subspace of algebraic functions from K to $\mathrm{Hom}_{\mathfrak{p}}(\mathcal{U}(\mathfrak{g}), V_{[P\cap K]})$ (using the definition of cohomological induction),

$$\Psi(f) = \Big[k \mapsto [\xi \mapsto (\xi \cdot k \cdot f)(1)]\Big], \quad k \in K, \xi \in \mathfrak{g}. \tag{2.33}$$

Also, by [Hec+, 6.3(i)], $\mathfrak{g}$ is spanned by $\mathfrak{k}$ and $\mathfrak{p}$, $\mathrm{Hom}_{\mathfrak{p}}(\mathcal{U}(\mathfrak{g}), V_{[P\cap K]}) \cong \mathrm{Hom}_{\mathfrak{p}\cap\mathfrak{k}}(\mathcal{U}(\mathfrak{k}), V_{[P\cap K]})$. Under this identification, we only need $\xi \in \mathfrak{k}$ in the above formula. This will be used in the following proof.

Lemma 2.34. *The map* $\Psi : \left(\mathrm{Ind}_{P_0}^{G_0} V\right)_{[K]} \to I_{(\mathfrak{p}, P\cap K)}^{(\mathfrak{g},K)} V_{[P\cap K]}$ *is injective.*

Proof. Suppose $f \in \left(\mathrm{Ind}_{P_0}^{G_0} V\right)_{[K]}$ is sent to 0. By the explicit description of of $\Psi(f)$ above, this means that for any $\xi \in \mathfrak{k}$, the map $k \mapsto (\xi \cdot k \cdot f)(1)$ is zero. On the other hand, as ξ ranges over a basis of $\mathfrak{k}_0$,

$$K_0 \longrightarrow \mathbb{C}, \quad k \mapsto (\xi \cdot k \cdot f)(1) = \frac{d}{dt} f\left(e^{t\xi} k\right)\Big|_{t=0} \tag{2.35}$$

are the first order coefficients of the Taylor expansion of $f|_{K_0}$. Since f is K-finite, $f|_{K_0}$ is real-analytic. Hence these coefficients being zero for all ξ implies that $f|_{K_0}$ is identically zero. Since $G_0 = P_0 K_0$ (by Iwasawa decomposition) and f is left P_0-linear, $f = 0$. Thus Ψ is injective. ∎

We will show that Ψ is an isomorphism by looking at K-types.

Lemma 2.36. *For any finite dimensional representation* η *of* K,

$$\mathrm{Hom}_K\left(\eta, \left(\mathrm{Ind}_{P_0}^{G_0} V\right)_{[K]}\right) \cong \mathrm{Hom}_{P\cap K}\left(\eta, V_{[P\cap K]}\right). \tag{2.37}$$

Proof. First, by left P_0-linearity and $G_0 = P_0 K_0$, $\mathrm{Ind}_{P_0}^{G_0} V = \mathrm{Ind}_{P_0\cap K_0}^{K_0} V$ as representations of K_0. Hence

$$\mathrm{Hom}_K\left(\eta, \left(\mathrm{Ind}_{P_0}^{G_0} V\right)_{[K]}\right) \cong \mathrm{Hom}_K\left(\eta, \mathrm{Ind}_{P_0}^{G_0} V\right) \cong \mathrm{Hom}_K\left(\eta, \mathrm{Ind}_{P_0\cap K_0}^{K_0} V\right) \tag{2.38}$$

(where the first equality is because any image of η in $\mathrm{Ind}_{P_0}^{G_0} V$ is automatically K-finite). On the other hand, by standard Frobenius reciprocity,

$$\mathrm{Hom}_K\left(\eta, \mathrm{Ind}_{P_0\cap K_0}^{K_0} V\right) \cong \mathrm{Hom}_{P\cap K}(\eta, V) \cong \mathrm{Hom}_{P\cap K}\left(\eta, V_{[P\cap K]}\right). \tag{2.39}$$

The desired equality is then obtained by combining these identities. ∎

Lemma 2.40.

$$\mathrm{Hom}_K\left(\eta, I_{(\mathfrak{p}, P\cap K)}^{(\mathfrak{g},K)} V_{[P\cap K]}\right) \cong \mathrm{Hom}_{P\cap K}\left(\eta, V_{[P\cap K]}\right). \tag{2.41}$$

Proof. Consider the following diagram

$$
\begin{tikzcd}
\mathrm{Mod}(\mathfrak{p}, P \cap K) \arrow[rr, "I^{(\mathfrak{g},K)}_{(\mathfrak{p},P\cap K)}"] \arrow[dd, "\mathrm{For}^{(\mathfrak{p},P\cap K)}_{(\mathfrak{p}\cap\mathfrak{k},P\cap K)}"'] \arrow[dr, "\mathrm{pro}^{(\mathfrak{g},P\cap K)}_{(\mathfrak{p},P\cap K)}"'] & & \mathrm{Mod}(\mathfrak{g}, K) \arrow[dd, "\mathrm{For}^{(\mathfrak{g},K)}_{(\mathfrak{k},K)}"] \\
& \mathrm{Mod}(\mathfrak{g}, P \cap K) \arrow[ur, "\Gamma^K_{P\cap K}"'] \arrow[dd, "\mathrm{For}^{(\mathfrak{g},P\cap K)}_{(\mathfrak{k},P\cap K)}"'] & \\
\mathrm{Mod}(\mathfrak{p} \cap \mathfrak{k}, P \cap K) \arrow[dr, "\mathrm{pro}^{(\mathfrak{k},P\cap K)}_{(\mathfrak{p}\cap\mathfrak{k},P\cap K)}"'] \arrow[rr, bend right=70, "I^{(\mathfrak{k},K)}_{(\mathfrak{p}\cap\mathfrak{k},P\cap K)}"'] & & \mathrm{Mod}(\mathfrak{k}, K) \\
& \mathrm{Mod}(\mathfrak{k}, P \cap K) \arrow[ur, "\Gamma^K_{P\cap K}"'] &
\end{tikzcd}
\tag{2.42}
$$

The restriction of $I^{(\mathfrak{g},K)}_{(\mathfrak{p},P\cap K)} V_{[P\cap K]}$ to K is then the image of $V_{[P\cap K]}$ from $\mathrm{Mod}(\mathfrak{p}, P \cap K)$ to $\mathrm{Mod}(\mathfrak{k}, K)$.

The triangles on the top and bottom commute by definition of I. For the middle-right square, the left-adjoints to the two paths are respectively $\mathrm{For}^{(\mathfrak{g},K)}_{(\mathfrak{g},P\cap K)} \circ \mathrm{ind}^{(\mathfrak{g},K)}_{(\mathfrak{k},K)}$ and $\mathrm{ind}^{(\mathfrak{g},P\cap K)}_{(\mathfrak{k},P\cap K)} \circ \mathrm{For}^{(\mathfrak{k},K)}_{(\mathfrak{k},P\cap K)}$, which are both equal to $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} -$. Hence they agree. Therefore the middle-right square commutes. For the middle-left square, the top path is the functor

$$\mathrm{For}^{(\mathfrak{g},P\cap K)}_{(\mathfrak{k},P\cap K)} \mathrm{pro}^{(\mathfrak{g},P\cap K)}_{(\mathfrak{p},P\cap K)}(-) = \mathrm{For}^{(\mathfrak{g},P\cap K)}_{(\mathfrak{k},P\cap K)} \mathrm{Hom}_{\mathfrak{p}}(\mathcal{U}(\mathfrak{g}), (-))_{[P\cap K]}. \tag{2.43}$$

Since $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, this is equal to

$$\mathrm{Hom}_{\mathfrak{p}\cap\mathfrak{k}}(\mathcal{U}(\mathfrak{k}), \mathrm{For}^{(\mathfrak{p},P\cap K)}_{(\mathfrak{p}\cap\mathfrak{k},P\cap K)}(-))_{[P\cap K]} = \mathrm{pro}^{(\mathfrak{k},P\cap K)}_{(\mathfrak{p}\cap\mathfrak{k},P\cap K)} \mathrm{For}^{(\mathfrak{p},P\cap K)}_{(\mathfrak{p}\cap\mathfrak{k},P\cap K)}(-) \tag{2.44}$$

which is the bottom path. Hence the middle-left square commutes.

Therefore, the restriction of $I^{(\mathfrak{g},K)}_{(\mathfrak{p},P\cap K)} V_{[P\cap K]}$ to K is isomorphic to $I^{(\mathfrak{k},K)}_{(\mathfrak{p}\cap\mathfrak{k},P\cap K)} V_{[P\cap K]}$ and hence

$$\mathrm{Hom}_K\left(\eta, I^{(\mathfrak{g},K)}_{(\mathfrak{p},P\cap K)} V_{[P\cap K]}\right) \cong \mathrm{Hom}_K\left(\eta, I^{(\mathfrak{k},K)}_{(\mathfrak{p}\cap\mathfrak{k},P\cap K)} V_{[P\cap K]}\right) \tag{2.45}$$

which equals

$$\mathrm{Hom}_{P\cap K}\left(\eta, V_{[P\cap K]}\right) \tag{2.46}$$

by adjointness of I with forgetful functor. ∎

Corollary 2.47. *The map* $\Psi : \left(\mathrm{Ind}^{G_0}_{P_0} V\right)_{[K]} \to I^{(\mathfrak{g},K)}_{(\mathfrak{p},P\cap K)} V_{[P\cap K]}$ *is an isomorphism.*

Proof. By 2.34 Ψ is injective. Therefore Ψ is an isomorphism if and only if for each K-type has the same multiplicity on both sides. Indeed, by 2.36 and 2.40, the multiplicity of η in both modules is equal to $\dim \mathrm{Hom}_{P\cap K}(\eta, V_{[P\cap K]})$. ∎

It was shown in a previous talk by Jack Cook that $I^{(\mathfrak{g},K)}_{(\mathfrak{p},P\cap K)}$ is exact, although by previous discussion (2.29) we have no need of this.

3. Comparison of classifications

Recall that $E = (T_x\tau^L)^\vee \otimes_{\mathbb{C}} T_x\omega_X$. Let U be the representation of M_0 whose $P_0 \cap K_0$-finite vectors equals $RI^{(\mathfrak{m},P\cap K)}_{(\mathfrak{d},T)} E|_{(\mathfrak{d},T)}$, a (limit of) discrete series representation of M_0. Therefore the representation V in §2.2 is equal to $U \boxtimes E|_{\mathfrak{a}}$. Combining results in the previous section, we obtain

$$\Gamma(X, \mathcal{I}(Q,\tau))^\vee \cong \left(\mathrm{Ind}^{G_0}_{P_0}(U \otimes E|_{\mathfrak{a}})\right)_{[K_0]}. \tag{3.1}$$

Using this, we finally obtain the following comparison of geometric classification and Langlands classification.

Comparison Theorem 3.2. *Let*

- λ *be strongly anti-dominant,*
- Q *a* K*-orbit on* X*,*
- $x_Q \in Q$*,*
- $\mathfrak{c} \subseteq \mathfrak{b}_{x_Q}$ *a* σ*-stable Cartan subalgebra,*
- τ *a irreducible* K*-homogeneous connection on* Q *compatible with* $\lambda + \rho$*,*
- $w \in W$ *such that* $\Sigma^+_{Q,w} \cap \sigma_Q(\Sigma^+_{Q,w}) = \varnothing$ *and* $\Sigma^+_{Q,w} \cup \sigma_Q(\Sigma^+_{Q,w}) = D_+(Q)$*,*
- Q^L *the Langlands orbit attached to* $\mathfrak{c}$ *determined by* $w^{-1}\Sigma^+_Q =: \Sigma^+_{Q^L}$*,*
- τ^L *as defined in 1.43,*
- $x \in Q^L$ *such that* $\mathfrak{b}_x$ *contains* $\mathfrak{c}$*,*
- $E = (T_x\tau^L)^\vee \otimes_{\mathbb{C}} T_x\omega_X$*,*
- $\mathfrak{p} \supseteq \mathfrak{c}$ *a* σ*-split parabolic determined by* $\Sigma_{Q,I} \cup w^{-1}\Sigma^+_Q = \Sigma^+_{Q^L,I}$*,*
- $\mathfrak{p}_0 \subseteq \mathfrak{g}_0$ *the real form of* $\mathfrak{p}$ *and* $P_0 \subseteq G_0$ *the corresponding group, and*
- U *the representation of* M_0 *whose* $P_0 \cap K_0$*-finite vectors equals* $RI^{(\mathfrak{m},P\cap K)}_{(\mathfrak{d},T)} E|_{(\mathfrak{d},T)}$*, a (limit of) discrete series representation of* M_0 *(see the beginning of §2.1 for notations on subalgebras of* $\mathfrak{p}$*).*

Then

$$\Gamma(X, \mathcal{I}(Q,\tau))^\vee \cong \left(\mathrm{Ind}^{G_0}_{P_0}(U \otimes E|_{\mathfrak{a}})\right)_{[K_0]}. \tag{3.1}$$

If $\Gamma(X, \mathcal{L}(Q,\tau)) \neq 0$*, the irreducible admissible* $(\mathfrak{g}, K)$*-module* $\Gamma(X, \mathcal{L}(Q,\tau))^\vee$ *corresponds to the Langlands datum*

$$(P_0, U, E|_{\mathfrak{a}}) \tag{3.3}$$

in Knapp-Zuckerman's Langlands classification [KZ77, Theorem 5].

Here $E|_{\mathfrak{a}} = -s_{x_Q}(\lambda - w^{-1}\rho)|_{\mathfrak{a}} = -s_x(w\lambda - \rho)|_{\mathfrak{a}}$ as calculated in the discussion preceding 2.21.

Proof. Since $\mathcal{L}(Q,\tau)$ is the unique irreducible submodule of $\mathcal{I}(Q,\tau)$, if $\Gamma(X, \mathcal{L}(Q,\tau)) \neq 0$, it is the unique irreducible submodule of $\Gamma(X, \mathcal{I}(Q,\tau))$. Taking contragradient, $\Gamma(X, \mathcal{L}(Q,\tau))^\vee$ is the unique irreducible quotient of $\Gamma(X, \mathcal{I}(Q,\tau))^\vee$ and, by (3.1), the latter is a parabolically induced module from a limit of discrete series of M_0 and a character of A_0.

To fit this into Langlands classification, it remains to show that the A_0-character $E|_{\mathfrak{a}} = -s_{x_Q}(\lambda - w^{-1}\rho)|_{\mathfrak{a}}$ is strongly dominant with respect to the set of restricted roots in $\mathfrak{p}_0$. The set of roots in $\mathfrak{p}_0$ restricted to $\mathfrak{a}$ are either real roots α in $\Sigma^+_{Q^L} = w^{-1}\Sigma^+_Q$ or $\alpha - \sigma_{Q^L}\alpha$ for complex roots α in $\Sigma^+_{Q^L} = w^{-1}\Sigma^+_Q$. In the latter case, note that $\alpha - \sigma_{Q^L}\alpha$ is in the $\mathbb{Q}_+$ span of real roots in $\Sigma^+_{Q^L}$.

Write

$$-s_{x_Q}(\lambda - w^{-1}\rho) = -s_{x_Q}(\lambda) + s_{x_Q}(w^{-1}\rho) = -s_{x_Q}(\lambda) + s_x(\rho). \tag{3.4}$$

$s_x(\rho)$ is strongly dominant with respect to $\Sigma^+_{Q^L}$. Consider $-s_{x_Q}(\lambda)$. Argue in the same way as in the last paragraph of the proof of 2.21 (with "imaginary" and "I" there replaced by "real" and

"$\mathbb{R}$"), one can show that

$$\Sigma^+_{Q^L,\mathbb{R}} \subseteq \Sigma^+_Q. \tag{3.5}$$

Since $s_{x_Q}(\lambda)$ is strongly antidominant w.r.t. roots in the second set,

$$\operatorname{Re} \alpha^\vee(-s_{x_Q}(\lambda)) \geqslant 0 \tag{3.6}$$

whenever α belongs to the first set. The same holds if α is replaced by anything in the $\mathbb{Q}_+$ span of $\Sigma^+_{Q^L,\mathbb{R}}$. Therefore

$$\operatorname{Re} \left(\alpha^\vee + (-\sigma_{Q^L}\alpha)^\vee\right)(-s_{x_Q}(\lambda)) \geqslant 0 \tag{3.7}$$

for any complex roots α in $\Sigma^+_{Q^L}$. Thus $-s_{x_Q}(\lambda)|_{\mathfrak{a}}$, and hence $-s_{x_Q}(\lambda - w^{-1}\rho)|_{\mathfrak{a}}$ is strongly dominant for all restricted roots in $\mathfrak{p}_0$, completing the proof. ∎

Remark 3.8 (Singular infinitesimal character). When λ is singular, it can happen that the global section of a standard $\mathcal{D}$-module equals the direct sum of global sections of other standard $\mathcal{D}$-modules. The same thing happens for the standard representations in the Knapp-Zuckerman classification (because the identification $\Gamma(X, \mathcal{I}(Q,\tau))^\vee \cong \left(\operatorname{Ind}_{P_0}^{G_0}(U \otimes E|_{\mathfrak{a}})\right)_{[K_0]}$ does not rely on regularity of λ). In other words, there are more standard modules than there are irreducible modules. Therefore, to obtain a classification of irreducibles in terms of standards, one needs to discard redundant standard modules. Geometrically, [Hec+, 10.2] provides some clue on how to do this.

However, to obtain a classification of irreducibles, it is easier (at least on the geometric side) to simply classify irreducible $\mathcal{D}$-modules and classify those with zero global section, instead of insisting on classifying irredundant standard modules first. This is done in *op. cit.* 9.1.

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