GEOMETRIC AND LANGLANDS CLASSFICATION FOR $\mathbf{SL}(2,\mathbb{R})$

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These notes are written for the representation theory student seminar at University of Utah, Fall 2021, which aims to achieve the following

- Present the geometric classification of irreducible admissible representations of $\mathbf{SL}(2,\mathbb{R})$ and compare it to Langlands classification.
- Realize (non-unitary) principal series representations of $\mathbf{SL}(2,\mathbb{R})$ geometrically and demonstrate/verify Casselman's Subrepresentation Theorem for $\mathbf{SL}(2,\mathbb{R})$.

The calculation is mostly based on [Hec+, 4].

If you find any mistakes in the notes, please let me know. It would be much appreciated.

Contents

General notations

- $G_0 = \mathbf{SL}(2,\mathbb{R})$,
- $B_0 = \left\{ \begin{pmatrix} * & * \\ & * \end{pmatrix} \in \mathbf{SL}(2,\mathbb{R}) \right\}$,
- $M_0 = \{\pm I \in \mathbf{SL}(2,\mathbb{R})\}$,
- $A_0 = \left\{ \begin{pmatrix} r & \\ & r^{-1} \end{pmatrix} \in \mathbf{SL}(2,\mathbb{R}) \mid r > 0 \right\}$,
- $N_0 = \left\{ \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \right\}$,
- $K_0 = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mid \theta \in \mathbb{R} \right\}$;
- $G = \mathbf{SL}(2,\mathbb{C})$,
- $B = \left\{ \begin{pmatrix} * & * \\ & * \end{pmatrix} \in \mathbf{SL}(2,\mathbb{C}) \right\}$,
- $M = \{\pm I \in \mathbf{SL}(2,\mathbb{C})\}$,
- $K = \left\{ \begin{pmatrix} * & \\ & * \end{pmatrix} \in \mathbf{SL}(2,\mathbb{C}) \right\}$;

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- $\mathfrak{g}_0 = \operatorname{Lie} G_0$, etc, and $\mathfrak{h} = \mathfrak{k}$;
- W Weyl group of the root system of $(\mathfrak{g}, \mathfrak{h})$;
- λ an element of $\mathfrak{h}^*$; $\theta = W \cdot \lambda$;
- $\mathcal{U}(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$; $\mathcal{Z}(\mathfrak{g})$ the center of $\mathcal{U}(\mathfrak{g})$; $S(\mathfrak{h})$ the symmetric algebra of $\mathfrak{h}$;
- $\chi_\lambda : \mathcal{Z}(\mathfrak{g}) \to \mathbb{C}$ the infinitesimal character determined by λ; $\mathcal{U}_\theta = \mathcal{U}(\mathfrak{g}) / \ker \chi_\lambda \mathcal{U}(\mathfrak{g})$;
- $\rho \in \mathfrak{h}^*$ is the half sum of positive roots determined by $\mathfrak{b}$;
- $H_r = \begin{pmatrix} & -i \\ i & \end{pmatrix}$, $E_r = \frac{1}{2}\begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}$, $F_r = \frac{1}{2}\begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}$;
- $H = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$, $E = \begin{pmatrix} & 1 \\ 0 & \end{pmatrix}$, $F = \begin{pmatrix} & 0 \\ 1 & \end{pmatrix}$.

1. Principal series

We first look at what principal series are and their structure.

Definition 1.1. A **(non-unitary) principal series representation** is a representation (L^2, continuously, or smoothly) induced from an irreducible finite dimensional representation of a minimal parabolic subgroup.

Principal series are useful because of their structures are easier to understand and because of the following theorem.

Theorem 1.2 (Casselman's Subrepresentation Theorem)**.** *Any irreducible admissible representation of* $(\mathfrak{g}_0, K_0)$ *on a Banach space can be embedded (*$(\mathfrak{g}_0, K_0)$*-linearly) into a principal series.*

In the case of $G_0 = \mathbf{SL}(2, \mathbb{R})$, $B_0 = M_0 A_0 N_0$ is a minimal parabolic subgroup. An irreducible representation of B_0 necessarily takes the form $(\varepsilon \otimes \nu, \mathbb{C})$ where $\varepsilon = 0$ or 1 (identified with the trivial or the sign representation of M_0 by abuse of notation) and $\nu \in \mathbb{C}$ (identified with the weight $\nu : \mathfrak{a}_0 \cong \mathbb{R} \to \mathbb{C}$, $r \mapsto \nu r$) with

$$(\varepsilon \otimes \nu)(\pm I) = (\pm 1)^\varepsilon \text{ for } \pm I \in M_0, \tag{1.3}$$

$$(\varepsilon \otimes \nu)(a) = e^{\nu \log a} \text{ for } a \in A_0, \tag{1.4}$$

$$(\varepsilon \otimes \nu)(n) = 1 \text{ for } n \in N_0. \tag{1.5}$$

Its continuous induction to G_0 is

$$\operatorname{Ind}_{B_0}^{G_0}(\varepsilon \otimes \nu) = \{f : G_0 \to \mathbb{C}_{\varepsilon,\nu} \mid f \text{ is continuous; } \forall\, p \in B_0, f(pg) = (\varepsilon \otimes \nu)(p) \cdot f(g)\}. \tag{1.6}$$

This is also frequently denoted by $I_{B_0,\varepsilon,\nu-1}$ (the parameter $\nu - 1$ is what people called the *normalized* parameter).

Taking K_0-finite vectors, we obtain a $(\mathfrak{g}_0, K_0)$-module $\left(\operatorname{Ind}_{B_0}^{G_0}(\varepsilon \otimes \nu)\right)_{[K_0]}$, where the $\mathfrak{g}_0$-action is given by

$$(\xi \cdot f_n)(g) = \frac{d}{dt} f_n(g e^{t\xi}) \Big|_{t=0} \tag{1.7}$$

(convergence is automatic by K_0-finiteness). We want to describe the structure of this module.

Let η_n ($n \in \mathbb{Z}_{\geqslant 0}$) be the 1-dimensional representation of K_0 with

$$\eta_n \left(\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \right) \cdot v = e^{in\theta} v. \tag{1.8}$$

By Iwasawa decomposition $G_0 = N_0 A_0 K_0$, restriction to K_0 defines a K_0-equivariant linear isomorphism

$$\operatorname{Ind}_{B_0}^{G_0}(\varepsilon \otimes \nu) \cong \operatorname{Ind}_{M_0}^{K_0} \varepsilon, \tag{1.9}$$

and the right side encodes all information on K_0-types. Combined with Frobenius reciprocity,

$$\operatorname{Hom}_{K_0}(\eta_n, \operatorname{Ind}_{B_0}^{G_0}(\varepsilon \otimes \nu)) \cong \operatorname{Hom}_{K_0}(\eta_n, \operatorname{Ind}_{M_0}^{K_0} \varepsilon) \cong \operatorname{Hom}_{M_0}(\eta_n, \varepsilon). \tag{1.10}$$

$\operatorname{Hom}_{M_0}(\eta_n, \varepsilon)$ consists of linear maps $\varphi : \mathbb{C} \to \mathbb{C}$ satisfying $\varphi((-1)^n v) = (-1)^\varepsilon \varphi(v)$. This space is 1-dimensional if $n \equiv \varepsilon \bmod 2$ and is 0 otherwise. Hence,

$$\begin{aligned} \dim \left(\operatorname{Ind}_{B_0}^{G_0}(\varepsilon \otimes \nu)\right)_{[K_0],\eta_n} &= \dim \operatorname{Hom}_{K_0}(\eta_n, \left(\operatorname{Ind}_{B_0}^{G_0}(\varepsilon \otimes \nu)\right)_{[K_0]}) && (1.11)\\ &= \dim \operatorname{Hom}_{K_0}(\eta_n, \operatorname{Ind}_{B_0}^{G_0}(\varepsilon \otimes \nu)) && (1.12)\\ &= \begin{cases} 1 & n \equiv \varepsilon \bmod 2, \\ 0 & n \not\equiv \varepsilon \bmod 2, \end{cases} && (1.13) \end{aligned}$$

where the η_n subscript denotes the corresponding isotypic component. Let $\omega_{-n} \in \left(\operatorname{Ind}_{B_0}^{G_0}(\varepsilon \otimes \nu)\right)_{[K_0],\eta_n}$ be a nonzero vector. Then since $\left(\operatorname{Ind}_{B_0}^{G_0}(\varepsilon \otimes \nu)\right)_{[K_0]}$ is the sum of all K_0-types,

$$\{\omega_{-n} \mid n \in \eta + 2\mathbb{Z}\} \tag{1.14}$$

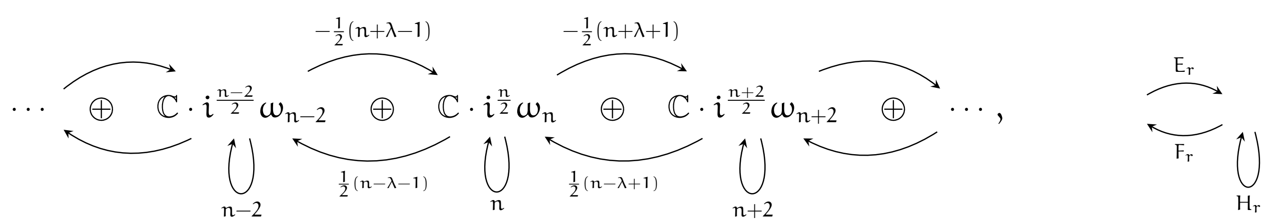
is a basis for $\left(\operatorname{Ind}_{B_0}^{G_0}(\varepsilon \otimes \nu)\right)_{[K_0]}$. Explicitly, we can take

$$\omega_{-n}|_{K_0} : K_0 \to \mathbb{C}, \quad \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mapsto e^{in\theta} \tag{1.15}$$

and extend it to G_0 by left N_0A_0-linearity. One computes (using (1.7) and Iwasawa decomposition) that

$$\begin{aligned} H_r \cdot i^{\frac{n}{2}} \omega_n &= n i^{\frac{n}{2}} \omega_n, && (1.16)\\ E_r \cdot i^{\frac{n}{2}} \omega_n &= -\frac{i}{2}(n+\nu) i^{\frac{n+2}{2}} \omega_{n+2}, && (1.17)\\ F_r \cdot i^{\frac{n}{2}} \omega_n &= -\frac{i}{2}(n-\nu) i^{\frac{n-2}{2}} \omega_{n-2}. && (1.18) \end{aligned}$$

To match the geometric calculation in §2, we set $\nu = \lambda + \rho = \lambda + 1$. Then the structure of $\left(\operatorname{Ind}_{B_0}^{G_0}(\varepsilon \otimes \nu)\right)_{[K_0]}$ can be described diagrammatically as



(1.19)

We also pre-compose the action $(\mathfrak{g}_0, K_0) \circlearrowright \left(\operatorname{Ind}_{B_0}^{G_0}(\varepsilon \otimes \nu)\right)_{[K_0]}$ with the complexification of the isomorphism

$$\mathbf{SU}(1,1) \xrightarrow{\sim} \mathbf{SL}(2,\mathbb{R}), \quad g \mapsto \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} g \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}^{-1} \tag{1.20}$$

which sends K to the complexification of K_0 and H to H_r, E to E_r and F to F_r. Then the action $\mathfrak{g} \circlearrowright \left(\operatorname{Ind}_{B_0}^{G_0}(\varepsilon \otimes \nu)\right)_{[K_0]}$ can be described by the same diagram except now arrows denote actions of E, F, H instead of E_r, F_r, H_r. From this, we can compute the action of the center $\mathcal{Z}(\mathfrak{g})$ by using the Casimir element

$$\Omega = H^2 - 2H + 4EF \tag{1.21}$$

(which generates $\mathcal{Z}(\mathfrak{g})$ as a $\mathbb{C}$-algebra). One checks that $\Omega \cdot \omega_n = (\lambda^2 - 1)\omega_n$. On the other hand, if χ_λ denotes the character on $\mathcal{Z}(\mathfrak{g})$ determined by λ, $\chi_\lambda(\Omega) = \lambda^2 - 1$ because, by the definition of χ_λ,

$$\Omega \xmapsto[\text{determined by } \{E,H,F\}]{\text{proj. to } \mathbb{C}\cdot H \text{ w.r.t. the PBW basis}} H^2 - 2H \xmapsto[\text{by } \rho]{\text{shift}} H^2 - 1 \xmapsto{\lambda} \lambda^2 - 1. \tag{1.22}$$

Hence $\left(\mathrm{Ind}_{B_0}^{G_0}(\varepsilon \otimes \nu)\right)_{[K_0]}$ has infinitesimal character χ_λ.

In §2 we will realize principal series and the Subrepresentation theorem via $\mathcal{D}$-modules on the flag variety.

2. Geometric classification

We turn to describing the geometric classification of irreducible admissible representations of $(\mathfrak{g}, K)$.

Let's first recall the general setting. Suppose $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$ is a complex semisimple Lie algebra, a Borel subalgebra, and a Cartan subalgebra, and $G \supset B \supset T$ algebraic groups with $\mathfrak{g}, \mathfrak{b}, \mathfrak{h}$ as respective Lie algebras. Let W be the Weyl group of the root system of $(\mathfrak{g}, \mathfrak{h})$. Consider the flag variety X of $\mathfrak{g}$. This is the variety of all Borel subalgebras of $\mathfrak{g}$; equivalently this is the variety G/B. For each $\lambda \in \mathfrak{h}^*$, there is a G-*homogeneous twisted sheaf of differential operators* ("htdo" for short) $\mathcal{D}_\lambda$ on X[1]. This parametrization is normalized so that when λ is integral, $\mathcal{D}_\lambda$ is the sheaf of differential operators on the homogeneous line bundle $\mathcal{O}_X(\lambda + \rho)$ (in particular $\mathcal{D}_{-\rho} = \mathcal{D}_X$ and $\mathcal{D}_\rho = \mathcal{D}_{\omega_X}$). If χ_λ is the infinitesimal character determined by λ and the Harish-Chandra homomorphism $\mathcal{Z}(\mathfrak{g}) \to S(\mathfrak{h})^W$, then

$$\Gamma(X, \mathcal{D}_\lambda) = \mathcal{U}(\mathfrak{g}) / \ker \chi_\lambda \mathcal{U}(\mathfrak{g}) \tag{2.1}$$

(due to Beilinson-Bernstein [BeBe81]; see also [Mil, 2.6]). Since $\chi_\lambda \subseteq \mathcal{Z}(\mathfrak{g})$ only depends on the W-orbit θ of λ, we denote $\mathcal{U}(\mathfrak{g})/\ker \chi_\lambda \mathcal{U}(\mathfrak{g})$ by $\mathcal{U}_\theta$.

Therefore we can *localize* a $\mathfrak{g}$-module to a $\mathcal{D}_\lambda$ on X, in the same way as localizing a module over a commutative ring R to produce a sheaf on $\operatorname{Spec} R$; conversely, given a $\mathcal{D}_\lambda$-module, its global section is a $\mathfrak{g}$-module. In nice cases, this is an equivalence of categories of modules on both sides. On one side we have $\mathrm{Mod}_{fg}(\mathcal{U}_\theta, K)$, the category of finitely generated $(\mathfrak{g}, K)$-modules with infinitesimal character χ_λ; on the other side we have $\mathrm{Mod}_{coh}(\mathcal{D}_\lambda, K)$, the category of coherent (i.e. locally finitely generated) $(\mathcal{D}_\lambda, K)$-modules. Here a $(\mathcal{D}_\lambda, K)$-module is a $\mathcal{D}_\lambda$-module with a *compatible* K-action, where compatibility means that the action of $\mathfrak{k} \subset \mathcal{D}_\lambda$ agrees with the action coming from differentiation of K-action).

In order to achieve equivalence, we need some conditions on λ. λ is said to be **antidominant** if $\alpha^\vee(\lambda) \notin \mathbb{Z}_{>0}$ for any positive root α, and **regular** if $\alpha^\vee(\lambda) \neq 0$ for any root α.

Theorem 2.2 (Beilinson-Bernstein; see also [Mil, 3.1])**.**

- *If λ is antidominant, localization is an equivalence of categories*

$$\mathrm{Mod}(\mathcal{U}_\theta, K) \cong \mathfrak{Q}\,\mathrm{Mod}_{coh}(\mathcal{D}_\lambda, K) \tag{2.3}$$

where $\mathfrak{Q}\,\mathrm{Mod}_{coh}(\mathcal{D}_\lambda, K)$ is the quotient of $\mathrm{Mod}_{coh}(\mathcal{D}_\lambda, K)$ by modules with no global sections.
- *If λ is antidominant regular, localization is an equivalence of categories*

$$\mathrm{Mod}(\mathcal{U}_\theta, K) \cong \mathrm{Mod}_{coh}(\mathcal{D}_\lambda, K). \tag{2.4}$$

[1] Here λ should really be an element of $\mathfrak{H}^*$ where $\mathfrak{H}$ is the *universal Cartan algebra* of $\mathfrak{g}$. Since we won't go into the precise construction of $\mathcal{D}_\lambda$'s, this won't make a difference later.

- *If* λ *is regular, derived localization is an equivalence of categories*
$$D(\mathcal{U}_\theta) \cong D(\mathcal{D}_\lambda) \tag{2.5}$$
where the left side the derived category of $\mathcal{U}_\theta$*-modules, and the right side is the derived category of quasicoherent* $\mathcal{D}_\lambda$*-modules.*

The quasi-inverse to these equivalences are given by the (derived) functor of taking global sections.

Therefore we can translate the study of $(\mathfrak{g}, K)$-modules with an infinitesimal character to $\mathcal{D}$-modules. Regarding irreducible $\mathcal{D}$-modules:

Theorem 2.6 (Beilinson-Bernstein; see also [Mil, §4.5])**.** *Irreducible coherent* $(\mathcal{D}_\lambda, K)$*-modules are parametrized by pairs* (Q, τ) *where* Q *is a* K*-orbit in* X *and* τ *is an irreducible* K*-homogeneous connection on* Q ***compatible with*** $\lambda + \rho$*. It is the unique irreducible submodule of the direct image of* τ *to* X*.*

Here, "τ compatible with $\lambda+\rho$" means that τ is a $(\mathcal{D}_\lambda|_{\{0\}}, K)$-module, where $\mathcal{D}_\lambda|_{\{0\}}$ is the pull-back of $\mathcal{D}_\lambda$ to $\{0\}$ along the inclusion map. Let $\mathcal{L}(Q,\tau)$ denote the irreducible module corresponding to (Q,τ) and $\mathcal{I}(Q,\tau)$ the direct image of τ. $\mathcal{I}(Q,\tau)$ is called a **standard module**.

Now we specify to $\mathbf{SL}(2,\mathbb{C})$.

Let $X = \mathbb{P}^1$. The representation $G = \mathbf{SL}(2,\mathbb{C}) \circlearrowright \mathbb{C}^2$ descends to a transitive action on X whose stabilizer at the image of $(1,0) \in \mathbb{C}^2$ is B. Hence the orbit map induces an isomorphism $G/B \xrightarrow{\sim} X$ and X is the flag variety of G. Let $0, \infty \in X$ denote the images of $(1,0), (0,1) \in \mathbb{C}^2$, respectively. Let $U_0 = X - \{\infty\}$ and $U_\infty = X - \{0\}$, both isomorphic to $\mathbb{A}^1$ as varieties with coordinates
$$x : U_0 \to \mathbb{C}, \quad (s,t) \mapsto \frac{t}{s}, \tag{2.7}$$
$$y : U_\infty \to \mathbb{C}, \quad (s,t) \mapsto \frac{s}{t}, \tag{2.8}$$
respectively. On a point z in the overlap $z \in U_0 \cap U_\infty = \mathbb{C}^*$, $x(z) = 1/y(z)$.

Let $\lambda, \rho \in \mathfrak{h}^*$ as before. We can describe local sections of $\mathcal{D}_\lambda$ explicitly using a trivialization over U_0 and U_∞, as follows.

Lemma 2.9. *There is a trivialization of* $\mathcal{D}_\lambda$ *on* X *described as follows.*

- *On* U_0*,*
$$E = -x^2\partial_x - (\lambda+\rho)(H)x, \tag{2.10}$$
$$H = 2x\partial_x + (\lambda+\rho)(H), \tag{2.11}$$
$$F = \partial_x. \tag{2.12}$$
(i.e. there is an isomorphism $\mathcal{D}_\lambda|_{U_0} \xrightarrow{\sim} \mathcal{D}_{U_0}$ *so that the composition* $\mathfrak{g} \to \Gamma(X, \mathcal{D}_\lambda) \to \Gamma(U_0, \mathcal{D}_\lambda) \xrightarrow{\sim} \Gamma(U_0, \mathcal{D}_{U_0})$ *sends* E, F, H *to the respective operators on the right hand side),*
- *on* U_∞*,*
$$E = \partial_y, \tag{2.13}$$
$$H = -2y\partial_y - (\lambda+\rho)(H), \tag{2.14}$$
$$F = -y^2\partial_y - (\lambda+\rho)(H)y \tag{2.15}$$
which, when further restricted to $\mathbb{C}^* = U_0 \cap U_\infty$*, equals (using* $y = x^{-1}$*,* $\partial_y = -x^2\partial_x$*)*
$$E = -x^2\partial_x, \tag{2.16}$$
$$H = 2x\partial_x - (\lambda+\rho)(H), \tag{2.17}$$
$$F = \partial_x - (\lambda+\rho)(H)x^{-1}; \tag{2.18}$$

- *and on $\mathbb{C}^*$ the transition map is given by (in the coordinate (x, ∂_x))*

$$\mathcal{D}_{U_0}|_{\mathbb{C}^*} \xrightarrow{\sim} \mathcal{D}_{U_\infty}|_{\mathbb{C}^*}, \quad \partial_x \mapsto \partial_x - (\lambda+\rho)(H)x^{-1}. \tag{2.19}$$

These are calculated using the definition of $\mathcal{D}_\lambda$ and the bracket relations between E, F, H. See [Hec+, 4] for details.

Now we look at the standard and irreducible $\mathcal{D}$-modules on X. K-orbits are $\{0\}$, $\{\infty\}$ and $\mathbb{C}^*$.

2.1. **$\mathcal{D}$-modules on closed orbits.** Let $i_0 : \{0\} \to X$ be the inclusion map. The pullback of $\mathcal{D}_\lambda$ to $\{0\}$ is denoted by $\mathcal{D}_\lambda^{i_0}$. Recall that, for a morphism $\varphi : G_1/S_1 \to G_2/S_2$ of homogeneous spaces, G_i-htdo's on G_i/S_i are parametrized by S_i-invariant elements in $\mathfrak{s}_i^*$, and for $\lambda \in (\mathfrak{s}_2^*)^{S_2}$, the pullback $\mathcal{D}^{\varphi}_{G_2/H_2,\lambda}$ of $\mathcal{D}_{G_2/H_2,\lambda}$ has parameter given by $\lambda|_{\mathfrak{s}_1}$. Applied to our situation, $\mathcal{D}_\lambda^{i_0}$ has parameter given by $\lambda|_{\mathfrak{h}}$. This means that the image of H under $\mathfrak{k} \to \mathcal{D}_\lambda^{i_0}$ is equal to $(\lambda+\rho)(H) = \lambda + 1$.

Let τ be an irreducible K-homogeneous connection on the orbit $\{0\}$. The stabilizer of the only point is K itself. So τ, viewed as an irreducible K-homogenous vector bundle, is simply an irreducible algebraic representation of K, which must be of the form $\mathbb{C}_\mu$ for some integral $\mu \in \mathfrak{k}^* = \mathfrak{h}^*$. On the other hand, τ is a $\mathcal{D}_\lambda^{i_0}$-module, so H acts on τ by $\lambda+\rho$. Hence $\lambda + \rho = \mu$, and λ must be integral for τ to exist.

Assuming integrality of λ, $\mathcal{I}(\{0\}, \lambda)$ can be computed explicitly by the definition of direct image functor: let $\mathfrak{m}_0 \subseteq \mathcal{O}_X$ be the ideal (sheaf) of functions vanishing on $\{0\}$, then

$$\mathcal{I}(\{0\}, \lambda) = i_{0,+}\mathbb{C}_{\lambda+\rho} \tag{2.1.1}$$
$$= i_{0,*}\left(\mathcal{D}_\lambda/\mathcal{D}_\lambda \mathfrak{m}_0 \otimes_{\mathbb{C}} \mathbb{C}\right) \tag{2.1.2}$$
$$= i_{0,*}\mathcal{D}_\lambda/\mathcal{D}_\lambda \mathfrak{m}_0. \tag{2.1.3}$$

On U_0 $\mathcal{D}_\lambda \cong \mathcal{D}_{U_0}$ has basis given by $\partial_x^m x^n$, $m, n \in \mathbb{Z}_{\geqslant 0}$. So

$$\mathcal{D}_\lambda/\mathcal{D}_\lambda \mathfrak{m}_0 = \operatorname{span}_{\mathbb{C}}\{1, \partial_x, \partial_x^2, \ldots\}. \tag{2.1.4}$$

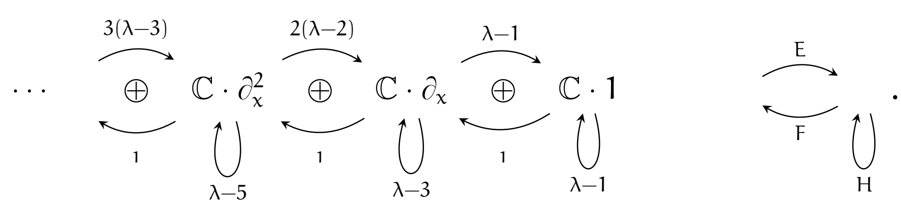
The Lie algebra elements acts as left multiplication by the operators given in the trivialization (2.10)-(2.12). So

$$H \cdot \partial_x^m = (2x\partial_x + (\lambda+1))\partial_x^m \tag{2.1.5}$$
$$= 2x\partial_x^{m+1} + (\lambda+1)\partial_x^m \tag{2.1.6}$$
$$= 2\partial_x^{m+1}x - 2(m+1)\partial_x^m + (\lambda+1)\partial_x^m \tag{2.1.7}$$
$$= 0 + (\lambda - 2m - 1)\partial_x^m \tag{2.1.8}$$
$$= (\lambda - 2m - 1)\partial_x^m. \tag{2.1.9}$$

Similarly

$$E \cdot \partial_x^m = m(\lambda - m)\partial_x^{m-1}, \tag{2.1.10}$$
$$F \cdot \partial_x^m = \partial_x^{m+1}. \tag{2.1.11}$$

Therefore the structure of $\Gamma(U, \mathcal{I}(\{0\}, \lambda))$ for any open set $U \ni 0$ can be described diagrammatically as



(2.1.12)

This is an irreducible $\mathcal{D}$-module because x sends $\mathbb{C}\cdot\partial_x^m$ to $\mathbb{C}\cdot\partial_x^{m-1}$ and ∂_x sends $\mathbb{C}\cdot\partial_x^m$ to $\mathbb{C}\cdot\partial_x^{m+1}$. This can also be seen using Kashiwara's theorem:

Theorem 2.1.13 (Kashiwara (see [Bor+87, VI.7]))**.** *If $\varphi : Z \hookrightarrow X$ is a closed immersion between smooth varieties and $\mathcal{D}$ a tdo on X, φ_+ is concentrated at degree 0 and $H^0\varphi_+$ is an equivalence of categories*

$$\mathrm{Mod}_{\mathrm{qcoh}}(\mathcal{D}^{\varphi}) \cong \mathrm{Mod}_{\mathrm{qcoh},Z}(\mathcal{D}) \tag{2.1.14}$$

where the subscript qcoh denotes quasi-coherence, and the subscript Z denotes modules supported in Z. The quasi-inverse is the functor $\varphi^!$ which takes sections supported in Z. This restricts to an equivalence of categories between coherent modules.

The global section $\Gamma(X,\mathcal{I}(\{0\},\lambda))$ is the Verma module $M(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_{\lambda-\rho}$.

A similar thing happen for the orbit at $\{\infty\}$, except we get a Verma module $\bar{M}(-\lambda)$ for the opposite Borel subalgebra.

Lemma 2.1.15. *$\mathcal{I}(\{0\},\lambda)$ and $\mathcal{I}(\{\infty\},\lambda)$ exist if and only if λ is integral. If this is the case,*

$$\Gamma(X,\mathcal{I}(\{0\},\lambda)) = M(\lambda), \text{ and} \tag{2.1.16}$$

$$\Gamma(X,\mathcal{I}(\{\infty\},\lambda)) = \bar{M}(-\lambda). \tag{2.1.17}$$

They are irreducible precisely when $\lambda \leqslant 0$. If $\lambda \geqslant 1$, they contain $M(-\lambda)$ and $\bar{M}(-\lambda)$ as irreducible submodules, respectively, and the quotients are finite dimensional irreducible representations.

2.2. **Modules on the open orbit.** Now let us look at the open orbit $\mathbb{C}^*$. The stablizer of a point in K is $M = \{\pm I\}$, so any the $(\lambda+\rho)$-compatibility condition is void because it is a requirement that boils down to the action of Lie algebra of the stablizer. Hence we are left to find irreducible K-homogeneous vector bundle on $\mathbb{C}^*$. They correspond to irreducible representations of M, which can only be $\{\mathrm{trv}, \mathrm{sgn}\}$. Let τ_ε denote the vector bundle corresponding to $\varepsilon \in \{\mathrm{trv}, \mathrm{sgn}\}$. As before, by abuse of notation we also view ε as either 0 or 1.

Recall that global section of τ_ε on $\mathbb{C}^*$ is given by induction:

$$\Gamma(\mathbb{C}^*,\tau_\varepsilon) = \mathrm{Hom}_M(K,\mathbb{C}_\varepsilon). \tag{2.2.1}$$

Let z be the coordinate on $K \cong \mathbb{C}^*$, this space is

$$\{f \in \Gamma(K,\mathcal{O}_K) \mid f(-z) = (-1)^\varepsilon f(z)\} = \begin{cases} \bigoplus_{m\in\mathbb{Z}} \mathbb{C}\cdot z^{2m} & \varepsilon = 0, \\ \bigoplus_{m\in\mathbb{Z}} \mathbb{C}\cdot z^{2m+1} & \varepsilon = 1. \end{cases} \tag{2.2.2}$$

The action of H on $\Gamma(\mathbb{C}^*,\tau_\varepsilon)$ is the left regular representation:

$$H\cdot z^k = kz^k. \tag{2.2.3}$$

However we want to view τ_ε as "functions" on $K/M = \mathbb{C}^*$. To do this, notice that inverse image along the quotient map

$$\begin{array}{ccc} K & \xrightarrow{\cong} & \mathbb{C}^*_z \\ \downarrow & & \downarrow \\ K/M & \xrightarrow{\cong} & \mathbb{C}^*_x \end{array} \tag{2.2.4}$$

is an injection sending x to z^2. Hence we can view

$$\Gamma(\mathbb{C}^*,\tau_0) = \Gamma(\mathbb{C}^*,\mathcal{O}_{\mathbb{C}^*}) = \mathrm{span}_{\mathbb{C}}\{x^p \mid p \in \mathbb{Z}\} \tag{2.2.5}$$

as $\mathcal{O}$-modules with H-action given by

$$H\cdot x^p = H\cdot z^{2p} = 2pz^{2p} = 2px^p = 2x\partial_x\cdot x^p, \tag{2.2.6}$$

i.e. H acts as $2x\partial_x \in \mathcal{D}_{\mathbb{C}^*}$. Following the same spirit, we view

$$\Gamma(\mathbb{C}^*, \tau_1) = \operatorname{span}_{\mathbb{C}}\{x^{p+\frac{1}{2}} \mid p \in \mathbb{Z}\} \tag{2.2.7}$$

where H acts as $2x\partial_x$. In this way τ_ε become $(\mathcal{D}_{\mathbb{C}^*}, K)$-modules. There is only one isomorphism class of K-htdo on $\mathbb{C}^*$ (because the parameterizing set is $\mathfrak{m}^* = 0$), and the isomorphism from the trivialization (2.10)-(2.12) is given explicitly by

$$\mathcal{D}_{U_0}|_{\mathbb{C}^*} \xrightarrow{\sim} \mathcal{D}_\lambda|_{\mathbb{C}^*} \xrightarrow{\sim} \mathcal{D}_{\mathbb{C}^*}, \quad \partial_x \mapsto \partial_x - \frac{\lambda+1}{2x} \tag{2.2.8}$$

under which

$$E = -x^2\partial_x - \frac{\lambda+1}{2}x, \tag{2.2.9}$$

$$H = 2x\partial_x, \tag{2.2.10}$$

$$F = \partial_x - \frac{\lambda+1}{2x}, \tag{2.2.11}$$

acting on $\Gamma(\mathbb{C}^*, \tau_\varepsilon)$ by the usual action of differential operators on functions:

$$E \cdot x^{n+\frac{\varepsilon}{2}} = -\left(n + \frac{\varepsilon+\lambda+1}{2}\right)x^{n+1+\frac{\varepsilon}{2}}, \tag{2.2.12}$$

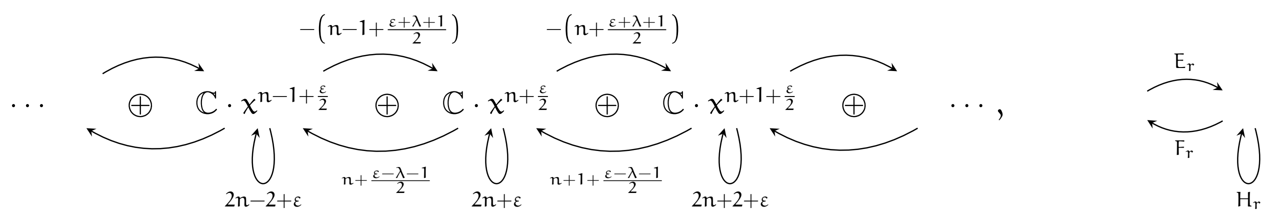
$$H \cdot x^{n+\frac{\varepsilon}{2}} = (2n+\varepsilon)x^{n+\frac{\varepsilon}{2}}, \tag{2.2.13}$$

$$F \cdot x^{n+\frac{\varepsilon}{2}} = \left(n + \frac{\varepsilon-\lambda-1}{2}\right)x^{n-1+\frac{\varepsilon}{2}}. \tag{2.2.14}$$

The space of global sections of the standard module is then

$$\Gamma(X, \mathcal{I}(\mathbb{C}^*, \tau_\varepsilon)) = \Gamma(X, i_{\mathbb{C}^*,*}\tau_\varepsilon) = \Gamma(\mathbb{C}^*, \tau_\varepsilon). \tag{2.2.15}$$

Using the explicit operators (2.2.9)-(2.2.11), it can be described diagrammatically as



(2.2.16)

Rewriting labels on the horizontal arrows by $-\left(n + \frac{\varepsilon+\lambda+1}{2}\right) = \frac{1}{2}((2n+\varepsilon)+\lambda+1)$ and $n + \frac{\varepsilon-\lambda-1}{2} = \frac{1}{2}((2n+\varepsilon)-\lambda-1)$ and compare with (1.19), we see that this is the principal series discussed in §1 via

$$\Gamma(X, \mathcal{I}(\mathbb{C}^*, \tau_\varepsilon)) \xrightarrow{\sim} I_{B_0,\varepsilon,\lambda}, \quad x^{n+\frac{\varepsilon}{2}} \mapsto i^{\frac{2n+\varepsilon}{2}}\omega_{2n+\varepsilon}. \tag{2.2.17}$$

(To be rigorous, I think we need to replace B_0 by its conjugate under (1.20)).

Let us analyze reducibility of this module. From the above diagram, it is clear that this is reducible if and only if either $\frac{\varepsilon+\lambda+1}{2} \in \mathbb{Z}$ or $\frac{\varepsilon-\lambda-1}{2} \in \mathbb{Z}$. This is equivalent to $\lambda+\varepsilon$ being an odd integer. If this happens for one ε, then it will fail for the other ε. Therefore, if λ is not integral, $\Gamma(X, \mathcal{I}(\mathbb{C}^*, \tau_\varepsilon))$ is irreducible; if λ is integral, one of $\Gamma(X, \mathcal{I}(\mathbb{C}^*, \tau_0))$, $\Gamma(X, \mathcal{I}(\mathbb{C}^*, \tau_1))$ is irreducible and the other is reducible.

The condition that $\lambda+\varepsilon$ is not an odd integer is called the **parity condition** in [Hec+].

Assuming λ to be integral, what is the unique irreducible submodule of the reducible one? By integrality of λ, the irreducible G-homogeneous connection $\mathcal{O}(\lambda+\rho)$ exists. It is a submodule of

$\mathcal{I}(\mathbb{C}^*,\tau_\varepsilon)$ if and only if there is a map into it. By adjunction of direct image and inverse image,

$$\mathrm{Hom}_{\mathcal{D}_\lambda}(\mathcal{O}(\lambda+\rho),\mathcal{I}(\mathbb{C}^*,\tau_\varepsilon)) = \mathrm{Hom}_{\mathcal{D}_{\mathbb{C}^*}}(\mathcal{O}(\lambda+\rho)|_{\mathbb{C}^*},\tau_\varepsilon) \tag{2.2.18}$$

$$\cong \mathrm{Hom}_M(\mathcal{O}(\lambda+\rho)(x_0),\tau_\varepsilon(x_0)). \tag{2.2.19}$$

where $x_0 \in \mathbb{C}^*$ and $-(x_0)$ takes geometric fiber at x_0. This is nonzero if λ has the correct parity. When this is the case, $\mathcal{O}(\lambda+\rho)$ embeds into one of $\mathcal{I}(\mathbb{C}^*,\tau_0)$, $\mathcal{I}(\mathbb{C}^*,\tau_1)$ and not into the other. In fact $\mathcal{O}(\lambda+\rho)$ embeds into the reducible one, because otherwise $\mathcal{O}(\lambda+\rho) = \mathcal{I}(\mathbb{C}^*,\tau_\varepsilon)$ (the irreducible one), which would imply

$$0 \neq i_0^*\mathcal{O}(\lambda+\rho) = i_0^*\mathcal{I}(\mathbf{C}^*,\tau_\varepsilon) = i_0^* i_{\mathbb{C}^*,*}\mathcal{O}_{\mathbb{C}^*} = 0 \tag{2.2.20}$$

as $\mathcal{O}$-modules, a contradiction.

Let $\mathcal{V}$ be the cokernel of the inclusion:

$$0 \longrightarrow \mathcal{O}(\lambda+\rho) \longrightarrow \mathcal{I}(\mathbb{C}^*,\tau_\varepsilon) \longrightarrow \mathcal{V} \longrightarrow 0. \tag{2.2.21}$$

To see what $\mathcal{V}$ is, look at the long exact sequence of derived pullback to $\{0\}$:

$$\begin{array}{l} 0 \longrightarrow i_0^!\mathcal{O}(\lambda+\rho) \longrightarrow i_0^!\mathcal{I}(\mathbb{C}^*,\tau_\varepsilon) \longrightarrow i_0^!\mathcal{V} \\ \longrightarrow i_0^*\mathcal{O}(\lambda+\rho) \longrightarrow i_0^*\mathcal{I}(\mathbb{C}^*,\tau_\varepsilon) \longrightarrow i_0^*\mathcal{V} \longrightarrow 0 \end{array} \tag{2.2.22}$$

$i_0^*\mathcal{I}(\mathbb{C}^*,\tau_\varepsilon) = 0$ by previous discussion. $i_0^*\mathcal{O}(\lambda+\rho) = \mathbb{C}$ as vector spaces. $i_0^!\mathcal{I}(\mathbb{C}^*,\tau_\varepsilon) = 0$ either by $\mathcal{D}$-module base change theorem, or by directly computing $\mathrm{Tor}^1_{\mathbb{C}[x]}(\mathbb{C}[x,x^{-1}],\mathbb{C}) = 0$. Therefore $Ri_0^!\mathcal{V} = \mathbb{C}$ as vector spaces. Similarly $Ri_\infty^!\mathcal{V} = \mathbb{C}$. Hence, by Kashiwara's theorem 2.1.13 $\mathcal{V} = \mathcal{I}(\{0\},\lambda) \oplus \mathcal{I}(\{\infty\},\lambda)$.

To summarize:

Lemma 2.2.23.

- $\Gamma(X,\mathcal{I}(\mathbb{C}^*,\tau_\varepsilon)) = I_{B_0,\varepsilon,\lambda}$ *is a principal series representation.*
- *If* λ *is not integral,* $\mathcal{I}(\mathbb{C}^*,\tau_\varepsilon)$ *is irreducible.*
- *If* λ *is integral and* λ *and* ε *satisfy the parity condition,* $\mathcal{I}(\mathbb{C}^*,\tau_\varepsilon)$ *is irreducible.*
- *If* λ *is integral and* λ *and* ε *fail the parity condition,* $\mathcal{I}(\mathbb{C}^*,\tau_\varepsilon)$ *is reducible and fits into the short exact sequence*

$$0 \longrightarrow \mathcal{O}(\lambda+\rho) \longrightarrow \mathcal{I}(\mathbb{C}^*,\tau_\varepsilon) \longrightarrow \mathcal{I}(\{0\},\lambda) \oplus \mathcal{I}(\{\infty\},\lambda) \longrightarrow 0. \tag{2.2.24}$$

2.3. **Geometric classification.** Invoking 2.2, we can obtain a classification of irreducible admissible $(\mathfrak{g},K)$-modules.

Every irreducible module has an infinitesimal character by Dixmier's lemma, whence lies in $\mathrm{Mod}_{fg}(\mathcal{U}_\theta,K)$ for some Weyl group orbit θ. Let $\lambda \in \theta$ be the unique strongly antidomiant element in θ, that is, $\mathrm{Re}\,\alpha^\vee(\lambda) \leqslant 0$ for the positive root α. When λ is identified with a complex number, this just means that $\mathrm{Re}\,\lambda \leqslant 0$. To avoid confusion, write $\mathcal{I}(\mathbb{C}^*,\tau_\varepsilon,\lambda)$ to indicate that this is an $\mathcal{D}_\lambda$-module. There are three cases.

- λ not integral. Then in particular λ is regular. Irreducible $(\mathcal{D}_\lambda,K)$-modules are $\mathcal{I}(\mathbb{C}^*,\tau_\varepsilon,\lambda)$ for both $\varepsilon = 0,1$ ($\mathcal{I}(\{0\},\lambda)$ and $\mathcal{I}(\{\infty\},\lambda)$ do not exist). Hence irreducible modules in $\mathrm{Mod}_{fg}(\mathcal{U}_\theta,K)$ are

$$\Gamma(X,\mathcal{I}(\mathbb{C}^*,\tau_0,\lambda)) = I_{B_0,0,\lambda}, \tag{2.3.1}$$

$$\Gamma(X,\mathcal{I}(\mathbb{C}^*,\tau_1,\lambda)) = I_{B_0,1,\lambda}, \tag{2.3.2}$$

which are irreducible principal series.

- $\lambda < 0$ integral. Irreducible $(\mathcal{D}_\lambda, K)$-modules are $\mathcal{I}(\{0\},\lambda)$, $\mathcal{I}(\{\infty\},\lambda)$, $\mathcal{O}(\lambda+\rho)$, and $\mathcal{I}(\mathbb{C}^*,\tau_\varepsilon,\lambda)$ where ε satisfies the parity condition with λ. Hence irreducible modules in $\mathrm{Mod}_{fg}(\mathcal{U}_\theta, K)$ are

$$\Gamma(X,\mathcal{O}(\lambda+\rho)) =: F_{\lambda+\rho}, \tag{2.3.3}$$
$$\Gamma(X,\mathcal{I}(\{0\},\lambda)) = M(\lambda) =: D_\theta^-, \tag{2.3.4}$$
$$\Gamma(X,\mathcal{I}(\{\infty\},\lambda)) = \bar{M}(-\lambda) =: D_\theta^+, \tag{2.3.5}$$
$$\Gamma(X,\mathcal{I}(\mathbb{C}^*,\tau_\varepsilon,\lambda)) = I_{B_0,\varepsilon,\lambda} \quad (\lambda, \varepsilon \text{ satisfy parity condition}). \tag{2.3.6}$$

  The first is a finite dimensional representation with lowest weight $\lambda+\rho$ by Borel-Weil theorem. The second and third are Verma modules. In the language of analytic representation theory, they are called the **discrete series** representations (their matrix coefficients are L^2 functions on G_0). The last one is an irreducible principal series.
- $\lambda = 0$. Irreducible $(\mathcal{D}_\lambda, K)$-modules are the same as in the previous case. However $\mathcal{O}(\rho)$ now has no global section. Hence irreducible modules in $\mathrm{Mod}_{fg}(\mathcal{U}_0, K)$ are

$$\Gamma(X,\mathcal{I}(\{0\},0)) = M(0) =: D_0^-, \tag{2.3.7}$$
$$\Gamma(X,\mathcal{I}(\{\infty\},0)) = \bar{M}(0) =: D_0^+, \tag{2.3.8}$$
$$\Gamma(X,\mathcal{I}(\mathbb{C}^*,\tau_\varepsilon,0)) = I_{B_0,0,0}. \tag{2.3.9}$$

  The first two are Verma modules. They are also classically called **limits of discrete series**. The third one is an irreducible principal series.

3. The Subrepresentation theorem

We want to show that every irreducible admissible module above embeds into a principal series representations.

The irreducible principal series embed into themselves.

For the finite dimensional representation, let $\lambda < 0$ be integral, and let ε fails the parity condition with λ. Apply $\Gamma(X,-)$ to the inclusion $\mathcal{O}(\lambda+\rho) \hookrightarrow \mathcal{I}(\mathbb{C}^*,\lambda,\tau_\varepsilon)$ produces the inclusion

$$F_{\lambda+\rho} \hookrightarrow I_{B_0,\varepsilon,\lambda} \tag{3.1}$$

into a reducible principal series.

For the discrete series, we have to look at domiant λ. Let $\lambda > 0$ be integral and ε failing the parity condition with λ. Take the long exact sequence of sheaf cohomologies on (2.2.24):

$$\begin{array}{l} 0 \longrightarrow \Gamma(X,\mathcal{O}(\lambda+\rho)) \longrightarrow \Gamma(X,\mathcal{I}(\mathbb{C}^*,\tau_\varepsilon)) \longrightarrow \Gamma(X,\mathcal{I}(\{0\},\lambda)) \oplus \Gamma(X,\mathcal{I}(\{\infty\},\lambda)) \\ \longrightarrow H^1(X,\mathcal{O}(\lambda+\rho)) \longrightarrow H^1(X,\mathcal{I}(\mathbb{C}^*,\tau_\varepsilon)) \longrightarrow H^1(X,\mathcal{I}(\{0\},\lambda)) \oplus H^1(X,\mathcal{I}(\{\infty\},\lambda)) \longrightarrow 0 \end{array}. \tag{3.2}$$

By Borel-Weil-Bott or (Serre duality) $\Gamma(X,\mathcal{O}(\lambda+\rho)) = 0$ and $H^1(X,\mathcal{O}(\lambda+\rho)) = \Gamma(X,\mathcal{O}(-\lambda+\rho)) = F_{-\lambda+\rho}$. $H^1(X,\mathcal{I}(\mathbb{C}^*,\tau_\varepsilon)) = 0$ because, if we write $\pi : X \to \{*\}$ for the morphism to a point,

$$H^1(X,\mathcal{I}(\mathbb{C}^*,\tau_\varepsilon)) = R^1\pi_* i_{\mathbb{C}^*,*}\tau_\varepsilon. \tag{3.3}$$

Since $\mathbb{C}^*$ is affine, $i_{\mathbb{C}^*}$ is an affine morphism. So $Ri_{\mathbb{C}^*} = i_{\mathbb{C}^*}$. Hence

$$H^1(X,\mathcal{I}(\mathbb{C}^*,\tau_\varepsilon)) = H^1R\pi_*Ri_{\mathbb{C}^*,*}\tau_\varepsilon = R^1(\pi\circ i_{\mathbb{C}^*})_*\tau_\varepsilon. \tag{3.4}$$

The map $\pi \circ i_{\mathbb{C}^*}$ is also affine, so $R^1(\pi \circ i_{\mathbb{C}^*})_* = 0$, and the claim follows. Therefore we obtain a short exact sequence

$$0 \longrightarrow I_{B_0,\varepsilon,\lambda} \longrightarrow M(\lambda) \oplus \bar{M}(-\lambda) \longrightarrow F_{-\lambda+\rho} \longrightarrow 0, \tag{3.5}$$

where $M(\lambda)$ and $\bar{M}(-\lambda)$ contains the discrete series D_θ^- and D_θ^+, respectively. If the image of $I_{B_0,\varepsilon,\lambda}$ in $M(\lambda) \oplus \bar{M}(-\lambda)$ does not contain D_θ^-, then $I_{B_0,\varepsilon,\lambda}$ will intersect trivially with D_θ^-, and D_θ^- will be mapped isomorphically into $F_{-\lambda+\rho}$. This is impossible by dimension consideration. Hence D_θ^- and similarly D_θ^+ are both contained in the principal series $I_{B_0,\varepsilon,\lambda}$. This completes the verification of the Subrepresentation theorem.

4. Comparison with Langlands classification

The Langlands classification states:

Theorem 4.1 (Langlands). *Let G_0 be a connected real semisimple Lie group with finite center, $G_0 = K_0A_0N_0$ a Iwasawa decomposition, $P_0 = M_0A_0N_0$ a minimal parabolic subgroup. Then irreducible admissible representations of $(\mathfrak{g}_0, K_0)$ are parameterized by triples*

$$(P'_0, \sigma, \lambda) \tag{4.2}$$

where $P'_0 = M'_0A'_0N'_0$ is a parabolic subgroup of G_0 containing P_0, σ is a tempered representation of M'_0, and $\nu \in (\mathfrak{a}'_0)^*$ is such that $\operatorname{Re} \alpha^\vee(\lambda) < 0$ for any positive restricted root α of $(\mathfrak{g}_0, \mathfrak{a}'_0)$ determined by N'_0. The irreducible representation corresponding to (P'_0, σ, λ) is the unique irreducible subrepresentation of the parabolically induced module $I_{P'_0,\sigma,\lambda} = \left(\operatorname{Ind}_{P'_0}^{G_0}(\sigma \otimes (\lambda + 1))\right)_{[K_0]}$.[2]

Let us verify this on $\mathbf{SL}(2, \mathbb{R})$.

Discrete series and the limits of discrete series are tempered. Hence they $\Gamma(X, \mathcal{I}(\{\bullet\}, \lambda)) = D_\theta^\pm$ (with $\bullet \in \{0, \infty\}$ and λ strongly antidominant) correspond to the triple $(G_0, D_\theta^\pm, 0)$.

The irreducible principal series $\Gamma(X, \mathcal{I}(\mathbb{C}^*, \tau_\varepsilon, \lambda)) = I_{B_0,\varepsilon,\lambda}$ (with λ strongly antidominant and ε satisfying the parity condition with λ) is itself parabolically induced. So it corresponds to $(B_0, \varepsilon, \lambda)$.

The finite dimensional representation $\Gamma(X, \mathcal{O}(\lambda + \rho)) = F_{\lambda+\rho}$ (for $\lambda \leqslant -1$ integral) embeds into a reducible principal series $I_{B_0,\varepsilon,\lambda}$ (with ε failing the parity condition with λ) as discussed in the previous section. Hence it corresponds to $(B_0, \varepsilon, \lambda)$.

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[2] Many books use the dual version where they require $\operatorname{Re} \alpha^\vee(\lambda) > 0$ instead. The irreducible module becomes the unique irreducible quotient of the induced module.