a) Construct the Taylor Polynomial $P_2(x)$ for $f(x) = \sqrt{x}$ using $x_0 = 1$ as the origin.

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f^{(n)}(x)$</td>
<td>$\sqrt{x} = x^{1/2}$</td>
<td>$f'(x) = \frac{1}{2}(x^{-1/2})$</td>
<td>$f''(x) = -\frac{1}{4}(x^{-3/2})$</td>
</tr>
<tr>
<td>$f^{(n)}(x_0) = f^{(n)}(1)$</td>
<td>$\sqrt{1} = 1$</td>
<td>$\frac{1}{2}(1)^{-1/2} = \frac{1}{2}$</td>
<td>$-\frac{1}{4}(1)^{-3/2} = -\frac{1}{4}$</td>
</tr>
<tr>
<td>$\frac{f^{(n)}(x_0)}{n!} (x-x_0)^n$</td>
<td></td>
<td>$\frac{1}{2}(x-1)^{0}$</td>
<td>$\frac{1}{2!}(x-1)^{1}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0!</td>
<td>1!</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>$\frac{1}{2}(x-1)$</td>
<td>$-\frac{1}{8}(x-1)^2$</td>
</tr>
</tbody>
</table>

$P_2(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2$

b) Drop the last term from $P_2(x)$ to get $P_1(x)$ and then estimate $\sqrt{2^7}$ to first order.

$P_1(x) = 1 + \frac{1}{2}(x-1)$

$\sqrt{2^7} \approx P_1(2) = 1 + \frac{1}{2}(2-1) = \frac{3}{2}$

Note: $\sqrt{x} = f(x) \approx P_1(x)$
c) Use the remainder \( R_n(x) \) to estimate the error. The sign of the error is irrelevant so take the absolute value.

\[
| R_n(x) | = \left| \frac{f^{(n+1)}(c)}{(n+1)!} (x-x_0)^{n+1} \right|
\]

\( c \) is between \( x_0 \) and \( x \)

For this problem...

\[ x_0 = 1 \leq c \leq 2 = x \] with \( n = 1 \)

\[ f''(x) = f''(x) = -\frac{1}{4} (x^{-3/2}) \quad \text{for} \quad f(x) = \sqrt{x} \]

\[
| R_1(x) | = \left| -\frac{1}{4 (2!)^3} (2-1)^2 \right| = \left| \frac{1}{8 \cdot c^{3/2}} \right|
\]

Notice that \( c = 1 \) gives the largest possible error (smallest denominator).

\[
| R_1(x) | \leq \left| -\frac{1}{8 (1)^{3/2}} (2-1)^2 \right| = \frac{1}{8}
\]

\[
\sqrt{2} = \frac{3}{2} \pm \frac{1}{8}
\]

Combining all results!

Notice that the error bar \((\pm \frac{1}{8})\) make the approximation an equality.