15.2 LINEARITY, INHOMOGENEOUS CASE (Theorem A, pg. 779)

consider: \( y'' + ay' + ay = k(x) \)

suppose: \( y_h \) is a solution for \( k(x) = 0 \)

where "h" stands for "homogeneous"
and \( y_p \) is a "particular" solution
for \( k(x) = 0 \)

then: \( y = y_h + y_p \) is also a solution

to see this, plug \( y = y_h + y_p \) into the
given differential equation and then
rearrange the pieces, as for the homogeneous

case on the previous page.

THREE STEPS (pg. 779)

consider \( y'' + ay' + ay = k(x) \)

1) find solution \( y_h \) for homogeneous
case (pretend \( k(x) \) equals zero)

2) guess a solution for the inhomogeneous
case (with \( A(x) \neq 0 \) included)

3) add \( y_p \) and \( y_h \) to get the general
solution \( y = y_h + y_p \)
Ex. 1 (pg. 380)

\[ y'' + y' - 2y = k(x) \text{ with } k(x) = 2x^2 - 10x + 3 \]

Solve homogeneous case

\[ y'' + y' - 2y = 0 \quad r \in \{-2, 1\} \]

\[ r^2 + r - 2 = 0 \quad y_h = C_1 e^{-2x} + C_2 e^x \]

\[ (r+2)(r-1) = 0 \]

Notice that \( k(x) \) is a polynomial.

Derivatives of polynomials give polynomials.

Guess: \( y_p = Ax^2 + Bx + C \)

\[ y_p' = 2Ax + B \]

\[ y_p'' = 2A \]

Plug in \( y_p \) for \( y \) in the given equation

\[ y'' + y' - 2y = 2x^2 - 10x + 3 \]

\[ 2A + (2Ax + B) - 2(Ax^2 + Bx + C) = 2x^2 - 10x + 3 \]

\( \frac{\text{1}}{\text{2}} \)

\[ (-2A)x^2 + (2A - 2B)x + (2A + B - 2C) = 2x^2 - 10x + 3 \]

Equate coefficients of \( x^2, x \) and \( x^0 \), \( = 1 \)

\[ (-2A) = 2 \Rightarrow A = -1 \]

\[ (2A - 2B) = -10 \quad \Rightarrow B = 4 \]

\[ (2A + B - 2C) = 3 \quad \Rightarrow C = -\frac{1}{2} \]
Ex. 1 (pg. 360, continued)

\[ y_p = Ax^2 + Bx + C = -x^2 + 4x - \frac{1}{2} \]

\[ y_h = C_1 e^{-2x} + C_2 e^x \]

add these to get the full solution

\[ y = y_h + y_p = C_1 e^{-2x} + C_2 e^x - x^2 + 4x - \frac{1}{2} \]

Ex. 2 (pg. 781)

\[ y'' + 2y' - 3y = k(x) \text{ with } k(x) = 8e^{3x} \]

Solve homogeneous case:

\[ y'' - 2y' - 3y = 0 \]

\( r^2 - 2r - 3 = 0 \) \( \{ r \in \{-1, 3\} \} \)

\[ y_h = C_1 e^{-x} + C_2 e^{3x} \]

Now solve inhomogeneous case

Notice that \( k(x) = 8e^{3x} \) is exponential
derivatives of exponentials give exponentials

Guess: \( y_p = Ae^{3x} \)

Problem: \( A = 8 \) gives part of \( y_h \) - duplication!

Modified guess: \( y_p = Axe^{3x} \) throw in a

factor of \( x \) \((e x^2 - x^3)\) to avoid duplication
Ex. 2 (pg. 781, continued)

Modified guess: \( y_p = Ax e^{3x} \)

\( y_p' = A(3e^{3x} + x(3e^{3x})) \) product rule!

\( y_p'' = A\left(9e^{3x} + 3e^{3x} + 9xe^{3x}\right) \)

\( 6e^{3x} \)

\( y'' - 2y' - 3y \)

\( Ae^{3x}\left(6 + 9x - 2(1 + 3x) - 3(x)\right) = 8e^{3x} \)

\( (e^{3x})4A = 8(e^{3x}) \Rightarrow A = 2 \)

Common factor

General solution:

\( y = y_h + y_p = C_1 e^{-x} + C_2 e^{3x} + 2xe^{3x} \)

Ex. 3 (pg. 781)

\( y'' - 2y' - 3y = \cos(2x) \) no \( k(x) = \cos(2x) \)

Solve homogeneous case:

\( y_h = C_1 e^{3x} + C_2 e^{-x} \) (see Ex. 2, p.781, in notes)

Notice \( k(x) = \cos(2x) \)

Repeated derivatives of \( \sin(x) \) and \( \cos(x) \) give \( \cos(x) \)
Ex. 3 (pg. 781, continued)

\[ y'' - 2y' - 3y = \cos(2x) \]

**Guess:** \[ y_p = A \sin(2x) + B \cos(2x) \]

\[ y_p' = 2A \cos(2x) - 2B \sin(2x) \]
\[ y_p'' = -4A \sin(2x) - 4B \cos(2x) \]

Factors of 2 in \( y' \) and 4 in \( y'' \) from chain rule!

\[ y'' - 2y' - 3y = \cos(2x) \]

\[ (-4A + 4B - 3A) \sin(2x) + ( -4B - 4A - 3B) \cos(2x) = (0) \sin(2x) + (1) \cos(2x) \]

\[ -7A - 7B = 0 \]
\[ -4A - 3B = 1 \]

\[ \begin{align*}
A &= \frac{4B}{7} \\
B &= \frac{-7}{65}
\end{align*} \]

Coefficients of \( \sin(2x) \) must be equal

Coefficients of \( \cos(2x) \) must be equal

Write down general solution:

\[ y = y_h + y_p = C_1 e^{3x} + C_2 e^{-x} - 4 \sin(2x) - \frac{7 \cos(2x)}{65} \]
**Educated Guessing (pg. 780)**

The guess for \( y_p \) depends on the form of \( r(x) \):

<table>
<thead>
<tr>
<th>( r(x) )</th>
<th>GUESS for ( y_p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b_m x^{m+...} + b_n x + b_0 )</td>
<td>( B_m x^{m+...} + B_n x + B_0 )</td>
</tr>
<tr>
<td>( b e^{ax} )</td>
<td>( B e^{ax} )</td>
</tr>
<tr>
<td>( b \cos(bx) + c \sin(bx) )</td>
<td>( B \cos(bx) + C \sin(bx) )</td>
</tr>
</tbody>
</table>

\( B_m, B_n, B_0, B \) & \( C \) are unknown constants.

**Modification:** If any term of \( r(x) \) solves the homogeneous equation then multiply the guess by \( x \) (or maybe \( x^2 \) or \( x^3 \)...).