15.2 Goal: solve $y'' + a_1 y' + a_0 y = k(x)$ by guessing

\[
\begin{align*}
    y''_h + a_1 y'_h + a_0 y_h &= 0 \\
    y''_p + a_1 y'_p + a_0 y_p &= k(x) \\
    y &= y_h + y_p
\end{align*}
\]

where $y$ is the "general" solution and the guess for "particular" solution $y_p$ depends on the form of $k(x)$.

there are rules for guessing.

Ex. 1 (pg. 380) polynomial guess STAGE I

$y'' + y' - 2y = k(x)$ with $k(x) = 2x^2 - 10x + 3$

derivatives of polynomials give polynomials

so make a polynomial guess

$y_p = Ax^2 + Bx + C$ (same order as $k(x)$)

$y'_p = 2Ax + B$

$y''_p = 2A$

plugging $y_p$ into the equation then group

terms with different powers of $x$
\[ y'' = 2x^2 - 10x + 3 \]

\[ 2A + (2Ax + B) - 2(Ax^2 + Bx + C) = 2x^2 - 10x + 3 \]

\[ 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \]

\[ x^2(-2A) + x(2A - 2B) + (2A + B - 2C) = x^2(2) + x(-10) + 3 \]

\[ 4 \quad 2 \quad 5 \quad 1 \quad 3 \quad 6 \]

**equate coefficients**

\[
\begin{align*}
(-2A) &= 2 \Rightarrow A = -1 \\
B &= 4 \\
(2A - 2B) &= -10 \\
(2A + B - 2C) &= 3
\end{align*}
\]

\[ y_p = -x^2 + 4x - \frac{1}{2} \text{ in a "particular" solution} \]

which has no arbitrary constants.

**adding solutions**  

**sum rule**

\[ y = y_h + y_p \Rightarrow y' = y'_h + y'_p \Rightarrow y'' = y''_h + y''_p \]

\[ y''_h + a_1 y'_h + a_0 y_h = 0 \]

\[ + y''_p + a_1 y'_p + a_0 y_p = k(x) \]

\[ (y''_h + y''_p) + a_1(y'_h + y'_p) + a_0(y_h + y_p) = 0 + k(x) \]

\[ y'' + a_1 y' + a_0 y = k(x) \]
ADDDING SOLUTIONS (CONTINUED)

So \( y = y_h + y_p \) is a solution of the
inhomogeneous equation but \( y_h \)
includes two arbitrary constants so
\( y \) includes two arbitrary constants.

Ex. 1 (pg. 960) STAGE II \( k(x) \)

Recall: \( y'' + y' - 2y = 2x^2 - 10x + 3 \Rightarrow y_p = -x + 4x - \frac{1}{2} \)

Imagine \( k(x) = 0 \) to get homogeneous case.

\( y'' + y' - 2y = 0 \) guess \( y = e^{rx} \)

\[
\begin{align*}
  r^2 + r - 2 &= 0 \\
  \alpha &= 1, \quad b = 1, \quad c = -2
\end{align*}
\]

\[
\begin{align*}
  ac &= \frac{b^2 - 4ac}{2a} \\
  r &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(-2)}}{2} = \frac{-1 \pm 3}{2}
\end{align*}
\]

\( y_h = C_1 e^{-4x} + C_2 e^{3x} \) solves the homogeneous
equation; \( y = y_h + y_p \) gives the most
general solution.

\[
y = C_1 e^{-4x} + C_2 e^{3x} - x + 4x - \frac{1}{2} \]

with two arbitrary constants.
15.2 Goal: Solve $y'' + ay' + by = R(x)$ by guesswork.

The guess for "particular" solution $y_p$ depends on the form of $R(x)$.

- $R(x)$: \( a(x) \)

- \( bmx^n + \ldots + b_0 \): \( Bmx^n + \ldots + B_0 \) \( \text{TRY: 3} \)

- \( be^{ax} \): \( Be^{ax} \) \( \text{TRY: 5} \)

- \( b \cos(\beta x) + c \sin(\beta x) \): \( B \cos(\beta x) + C \sin(\beta x) \) \( \text{TRY: 9} \)

\( b, B, b_0, B_0, b \) and \( c \) are unknown constants.

Modification: if any term of \( R(x) \) solves the homogeneous \((R(x) = 0)\) equation, then multiply the guess by \( x \) (or maybe \( x^2 \), \( x^3 \), \ldots, as required).

**Note:** \( \frac{d}{dx} \cos(2x) = -2 \sin(2x) \)

**Ex. 3** (no. 781) trig. guess \( u_5'' - 2u_5' + u_5 = R(x) \) with \( R(x) = \cos(2x) \)

Step one: solve the homogeneous equation \((R(x) = 0)\)

\( \frac{d^2}{dx^2} - 2 \frac{dy}{dx} - 3y = 0 \)

\( \downarrow \downarrow \downarrow \)

\( r^2 - 2r - 3 = 0 \)

\( (r + 1)(r - 3) = 0 \Rightarrow r \in \{-1, 3\} \)

\( y_h = C_1 e^{-x} + C_2 e^{3x} \) (for the case of real distinct roots)
Step two: guess a solution for $k(x) 
eq 0$

Note $h(x) = \cos(2x)$ so $y_p = A\sin(2x) + B\cos(2x)$

In the standard guess (since derivatives of $\sin(x)$ and $\cos(x)$ can each give $\cos(x)$)

Step three: consider modification

No modification required (since no term of $y_p$ duplicates $e^{-x}$ or $e^{3x}$ from $y_h$)

Step four: plug guess for $y_p$ into equation

and group terms

\[-3y_p = -3(A\sin(2x) + B\cos(2x))\]
\[-2y''_p = -2(2A\cos(2x) - 2B\sin(2x))\]
\[y''_p = +(-4A\sin(2x) - 4B\cos(2x))\]

\[(0)\sin(2x) + (1)\cos(2x) = \sin(x)(-3A + 4B - 4A) + \cos(x)(-3B - 4A - 4B)\]

Step five: equate coefficients

$\sin(2x)$: $-7A + 4B = 0$  \[\rightarrow\]  $\cos(2x)$: $-7B - 4A = 1$

$A = \frac{4B}{7}$  \[\rightarrow\]  $-7B - \frac{16B}{7} = 1$

$A = -\frac{4}{65}$  \[\rightarrow\]  $B = -\frac{7}{65}$

$y_p = -\frac{4}{65}\sin(2x) - \frac{7}{65}\cos(2x)$ is a solution
Step six: add $y_h$ and $y_p$

\[ y = y_h + y_p = C_1 e^{-x} + C_2 e^{3x} - \left( \frac{4}{65} \right) \sin(2x) - \left( \frac{7}{65} \right) \cos(2x) \]

This is the general solution.

Ex. 2 (pg. 78) modified guess

\[ y'' - 2y' - 3y = k(x) \quad \text{with} \quad k(x) = 8e^{3x} \]

Step one: solve the homogeneous equation ($k(x) = 0$)

Identical to Ex. 3

\[ y_h = C_1 e^{-x} + C_2 e^{3x} \quad \Rightarrow \quad y''_h - 2y'_h - 3y_h = 0 \]

Step two: guess a solution for $k(x) \neq 0$

Note $k(x) = 8e^{3x}$ so $y_p = Ae^{3x}$ is the standard guess (since derivatives of exponentials are exponentials).

Step three: consider modification

Requires modification (since $y_p = Ae^{3x}$ matches the second term of $y_h = C_1 e^{-x} + C_2 e^{3x}$)

Add a factor of $x$ to get $y_p = Ax e^{3x}$
Step four: plug guess for $y_p$ into equation and group terms

$$-3y_p = -3(Axe^{3x})$$
$$-2y_p' = -2(A(1)e^{3x} + 3Ax e^{3x}) \text{ product rule}$$
$$y_p'' = + (3Ae^{3x} + 3A(1)e^{3x} + 3xe^{3x})$$

$$8e^{3x} = Ae^{3x}(-3x - 2 - 6x + 3 + 3 + 9x) = 4Ae^{3x}$$

Step five: equate coefficients

$$8e^{3x} = 4Ae^{3x} \Rightarrow A = 2$$

So $y_p = 2xe^{3x}$ is a solution.

Step six: add $y_p$ and $y_h$

$$y = y_h + y_p = C_1e^{-x} + C_2e^{3x} + 2xe^{3x}$$

This is the general solution.