1) Testing Series

1.a) Test for Divergence

\[ \lim_{n \to \infty} a_n \neq 0 \Rightarrow \sum a_n \text{ divergent} \]

\[ \lim_{n \to \infty} a_n = 0 \text{ unconclusive!} \]

This can be quick and simple when the limit is easy (pg. 457, 9.3-15). DANGER!

Do not say "convergent" when the test is inconclusive. Try a different test.

1.b) Is the given series geometric?

\[ |r| < 1 \Rightarrow \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \text{ so convergent} \]

\[ |r| \geq 1 \Rightarrow \text{ divergent} \]

Learn to recognize a "disguised" geometric series (9.2-11). For example, the series can start at \( n=0 \) instead.

\[ |r| < 1 \Rightarrow \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \]
1.c) Is the given series a p-series?

\[ \sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{convergent for } p > 1 \\ \text{divergent for } p \leq 1 \end{cases} \]

(Ex. 3, Notes 98)

1.d.i) Limit Comparison Test

If the terms in a series have only powers of \( n \), like \( n^2, \sqrt{n} = n^{1/2} \) or \( 1 = n^0 \), for example, then keep the highest powers from the numerator and denominator of \( a_n \) to obtain \( b_n \) for comparison.

**Example:** \( a_n = \frac{n+3}{\sqrt{n^3+n}} \) becomes \( \frac{b_n = \frac{n}{\sqrt{n^3}}}{} \)

Keep \( n \) from numerator and \( n^3 \) from denominator, using \( \sqrt{n^3} = (n^3)^{1/2} = n^{3/2} \)

Now use the limit comparison test

(Notes 94, 9.4-3, 9.4-13)

When powers of \( n \) are mixed with other functions, try the next hint (1.d.ii) instead.
1.d.ii) Ordinary Comparison

For example, a smaller denominator gives a bigger fraction. Discarding or replacing some part of an can give a simpler $b_n$ for comparison.

**Example:** $a_n = \frac{1}{ne^n}, \frac{1}{n(n)} = \frac{1}{n^2} = b_n \quad (n \geq 1 \Rightarrow e^n > n)$

**Example:** $c_n = \frac{1}{n-e^{-n}} \Rightarrow \frac{1}{n} = d_n \quad (n \geq 1 \Rightarrow 1 > \frac{1}{e^n} > 0)$

Note that $\Sigma b_n$ and $\Sigma d_n$ are $p$-series and use ordinary comparison (pg. 469, 9.4-29).

$0 \leq a_n \leq b_n \& \Sigma b_n$ convergent $\Rightarrow \Sigma a_n$ converges

$0 \leq d_n \leq c_n \& \Sigma d_n$ divergent $\Rightarrow \Sigma c_n$ diverges

Note that limit comparison (hint 1.d.i) for $c_n$ and $d_n$ would be much handier.

1.e) If the terms of a series include $n!$ or $n$ as a power, like $2^n$, then
Try the absolute ratio test (notes 104, Ex. 4). This test is often used to find the "radius of convergence" (new notes I) and "convergence set" for a power series (new notes III, IV). The following cancelations often arise.

\[
\frac{x^{n+1}}{x^n} = \frac{(x^n)X}{x^n} = X \quad \text{AND} \quad \frac{n!}{(n+1)!} = \frac{n!}{(n+1)n!} = \frac{1}{n+1}
\]

DANGER! The absolute ratio test will fail if the terms in a series have only powers of n like \(n^2\) or \(\sqrt{n}\) when limit comparison (hint 1.d.i) should be used instead.

DANGER! Endpoints of the interval of convergence must be tested separately using some other test (new notes IV).

There is no point using the ordinary ratio test since absolute convergence implies ordinary convergence also.
1. (f) For series of the form $\sum (-1)^n a_n$ the alternating series test is an obvious choice (pg. 457, 9.5-1). This test can establish conditional convergence when absolute convergence fails. Know how to do this!

There will be NO question on the error for approximating an alternating series ($|S_n - \sum (-1)^n a_n| \leq a_{n+1}$).

1. (g) If a series $\sum a_n$ has terms of the form $a_n = f(n)$ with $f(x)$ 0 decreasing and $\int_{1}^{\infty} f(x)dx$ seems doable then consider the integral test (pg. 464) but this will NOT appear on your test.

2) Be prepared for the calculus (new notes V, VI) and algebra (new notes VII, 9.7-25, 9.8-3) of power series
Consider a geometric series with $a = 1$ and $x = r$ with $|x| < 1$ to ensure convergence, and take the derivative

$$(1-x)^{-1} = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{(1-x)^2} = \frac{d}{dx} (1-x)^{-1} = \sum_{n=0}^{\infty} \frac{d}{dx} x^n = \sum_{n=1}^{\infty} n x^{n-1}$$

Notice that final series starts at $n=1$ since the $n=0$ term of the original series is constant ($x^0 = 1$) which is destroyed by the derivative.

3) Be prepared to use a Taylor Polynomial (new notes XIII) to approximate a function (new notes XIV) and use the remainder term (new notes XIII) to estimate the error (new notes XIII, XIV).

The example of $f(x) = \sqrt{x}$ and $x_0 = 1$ is available on the web site. DANGER! you must understand the notation!