Generic Vanishing and Classification of Irregular Surfaces in Positive Characteristics

Yuan Wang

University of Utah

Nesin Mathematics Village, Turkey
Algebraic Surface

Definition

An *algebraic surface* or a *surface* is a nonsingular projective variety of dimension 2 over an algebraically closed field $k$.

Let $X$ be a surface.

1. **geometric genus** $p_g(X) := h^0(X, \omega_X)$.
2. **irregularity** $q(X) := h^1(X, \mathcal{O}_X)$.
3. **Euler characteristic** $\chi(\mathcal{O}_X) := h^0(X, \mathcal{O}_X) - h^1(X, \mathcal{O}_X) + h^2(X, \mathcal{O}_X)$.
4. **Kodaira dimension** $\kappa(X) := \max_{m \geq 1} \{ \dim(\phi_m(X)) \}$, where $\phi_m : X \dashrightarrow \mathbb{P}H^0(X, \omega_X^\otimes m)$ is the rational map induced by $|mK_X|$.
   - $\kappa(X) := -\infty$ if $|mK_X| = \emptyset$ for all $m \geq 1$. 

Yuan Wang  Generic vanishing & surface classification in char. $p$
For a surface $X$, $\kappa(X)$ can be $-\infty, 0, 1$ or $2$.

**Definition**

A surface is *minimal* if every birational transform $f : X \rightarrow X'$ to another nonsingular surface $X'$ is an isomorphism.

Let $X$ be a minimal surface.

1. $\kappa(X) = -\infty$: $\mathbb{P}^2$; ruled surface.
2. $\kappa(X) = 0$: Enriques surface; K3 surface; abelian surface; hyperelliptic surface; quasi-hyperelliptic surface (only exists in characteristic 2, 3).
3. $\kappa(X) = 1$: elliptic surface; quasi-elliptic surface (only exists in characteristic 2, 3).
4. $\kappa(X) = 2$: surface of general type.
Two important inequalities:

**Theorem (Castelnuovo’s inequality)**

Let $X$ be a minimal surface of general type in characteristic $0$, then $\chi(O_X) > 0$.

(also holds in characteristic $p \geq 11$, by Shepherd-Barron.)

**Theorem (Bogolomov-Miyaoka-Yau inequality)**

Let $X$ be a minimal surface of general type in characteristic $0$, then $K_X^2 \leq 9\chi(O_X)$.

(also holds for surfaces that lift to $W_2(k)$ in characteristic $p \geq 3$, by Langer.)
Classification of surfaces of general type

Characteristic 0: Let $X$ be a minimal surface of general type.

\[ \chi(O_X) = 1 \Rightarrow p_g(X) = q(X) \leq 4. \] (Debarre)

- $p_g(X) = q(X) = 4 : X = C_1 \times C_2$ where $g(C_1) = g(C_2) = 2$. (Beauville)
- $p_g(X) = q(X) = 3$: either
  - $X = \text{Sym}^{(2)}(C)$ where $g(C) = 3$, or
  - $X = (C_1 \times C_2)/(\sigma_1 \times \sigma_2)$ where $g(C_1) = 2$ with elliptic involution $\sigma_1$, and $g(C_2) = 3$ with free involution $\sigma_2$. (Hacon-Pardini; Pirola)
- $p_g(X) = q(X) \leq 2$: partially classified. (Zucconi)

$\chi(O_X) \geq 2$: very little is known.
Classification of surfaces of general type

Characteristic $p > 0$: ?
Let $X$ be a smooth complex projective variety of dimension $n$ and $a : X \rightarrow A$ the Albanese morphism of $X$.

For an integer $i \geq 0$ let $V^i(\omega_X)$ be the subvariety of Pic$^0(X)$ defined by

$$V^i(\omega_X) = \{ P \in \text{Pic}^0(X) | H^i(X, \omega_X \otimes P) \neq 0 \}$$

Theorem (Green & Lazarsfeld 1987)

$$\text{codim}(V^i(\omega_X), \text{Pic}^0(X)) \geq \dim(a(X)) - n + i.$$  

In particular if $X$ is of maximal Albanese dimension, then

$$\text{codim}(V^i(\omega_X), \text{Pic}^0(X)) \geq i.$$
In general generic vanishing fails in positive characteristics. A counter-example was given by Hacon-Kovács in dimension 3. However for surfaces we establish the following

Theorem (-)

Let $X$ be a smooth projective surface over an algebraically closed field $k$ of positive characteristic. If $X$ is of maximal Albanese dimension and lifts to $\text{W}_2(k)$ then $\text{codim}(V^i(\omega_X), \text{Pic}^0(X)) \geq i$.

Unknown if we do not assume liftability to $\text{W}_2(k)$. 
Classification of surfaces of general type

Theorem (-)

Let $X$ be a smooth minimal surface of general type over an algebraically closed field $k$. Denote the Albanese morphism of $X$ as $\alpha : X \to A$. Assume that

- $\text{char}(k) \geq 11$.
- $X$ is of maximal Albanese dimension, lifts to $W_2(k)$, its Picard variety has no supersingular factors.
- $\chi(\mathcal{O}_X) = 1$ and $\dim(A) = 4$.
- $\alpha$ is separable.

Then $X = C_1 \times C_2$ where $C_1$ and $C_2$ are smooth curves and $g(C_1) = g(C_2) = 2$. 

Yuan Wang

Generic vanishing & surface classification in char. $p$
Under the condition of the theorem, we actually have that $p_g(X) = q(X) = 4$.

Use of $\text{char}(k) \geq 11$: to show generic smoothness of a fibration $f : X \to C$ using a theorem of Tate.

We require that $a$ is separable only to guarantee that there exists a 2-form $\omega$ on $A$ such that $a^* \omega \neq 0$ (known for 1-form by Igusa).