Theorem 5.6 (Schwarz reflection principle) *Suppose that f is a holomorphic function in Ω^+ that extends continuously to I and such that f is real-valued on I. Then there exists a function F holomorphic in all of Ω such that $F = f$ on Ω^+.*

Proof. The idea is simply to define $F(z)$ for $z \in \Omega^-$ by

$$F(z) = \overline{f(\overline{z})}.$$

To prove that F is holomorphic in Ω^- we note that if $z, z_0 \in \Omega^-$, then $\overline{z}, \overline{z_0} \in \Omega^+$ and hence, the power series expansion of f near $\overline{z_0}$ gives

$$f(\overline{z}) = \sum a_n(\overline{z} - \overline{z_0})^n.$$

As a consequence we see that

$$F(z) = \sum \overline{a_n}(z - z_0)^n$$

and F is holomorphic in Ω^-. Since f is real valued on I we have $\overline{f(x)} = f(x)$ whenever $x \in I$ and hence F extends continuously up to I. The proof is complete once we invoke the symmetry principle.

5.5 Runge's approximation theorem

We know by Weierstrass's theorem that any continuous function on a compact interval can be approximated uniformly by polynomials.[4] With this result in mind, one may inquire about similar approximations in complex analysis. More precisely, we ask the following question: what conditions on a compact set $K \subset \mathbb{C}$ guarantee that any function holomorphic in a neighborhood of this set can be approximated uniformly by polynomials on K?

An example of this is provided by power series expansions. We recall that if f is a holomorphic function in a disc D, then it has a power series expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$ that converges uniformly on every compact set $K \subset D$. By taking partial sums of this series, we conclude that f can be approximated uniformly by polynomials on any compact subset of D.

In general, however, some condition on K must be imposed, as we see by considering the function $f(z) = 1/z$ on the unit circle $K = C$. Indeed, recall that $\int_C f(z)\, dz = 2\pi i$, and if p is any polynomial, then Cauchy's theorem implies $\int_C p(z)\, dz = 0$, and this quickly leads to a contradiction.

[4] A proof may be found in Section 1.8, Chapter 5, of Book I.

A restriction on K that guarantees the approximation pertains to the topology of its complement: K^c must be connected. In fact, a slight modification of the above example when $f(z) = 1/z$ proves that this condition on K is also necessary; see Problem 4.

Conversely, uniform approximations exist when K^c is connected, and this result follows from a theorem of Runge which states that for *any* K a uniform approximation exists by *rational functions* with "singularities" in the complement of K.[5] This result is remarkable since rational functions are globally defined, while f is given only in a neighborhood of K. In particular, f could be defined independently on different components of K, making the conclusion of the theorem even more striking.

Theorem 5.7 *Any function holomorphic in a neighborhood of a compact set K can be approximated uniformly on K by rational functions whose singularities are in K^c.*

If K^c is connected, any function holomorphic in a neighborhood of K can be approximated uniformly on K by polynomials.

We shall see how the second part of the theorem follows from the first: when K^c is connected, one can "push" the singularities to infinity thereby transforming the rational functions into polynomials.

The key to the theorem lies in an integral representation formula that is a simple consequence of the Cauchy integral formula applied to a square.

Lemma 5.8 *Suppose f is holomorphic in an open set Ω, and $K \subset \Omega$ is compact. Then, there exists finitely many segments $\gamma_1, \ldots, \gamma_N$ in $\Omega - K$ such that*

$$f(z) = \sum_{n=1}^{N} \frac{1}{2\pi i} \int_{\gamma_n} \frac{f(\zeta)}{\zeta - z}\, d\zeta \quad \textit{for all } z \in K. \tag{15}$$

Proof. Let $d = c \cdot d(K, \Omega^c)$, where c is any constant $< 1/\sqrt{2}$, and consider a grid formed by (solid) squares with sides parallel to the axis and of length d.

We let $\mathcal{Q} = \{Q_1, \ldots, Q_M\}$ denote the finite collection of squares in this grid that intersect K, with the boundary of each square given the positive orientation. (We denote by ∂Q_m the boundary of the square Q_m.) Finally, we let $\gamma_1, \ldots, \gamma_N$ denote the sides of squares in $\mathcal{Q}$ that do not belong to two adjacent squares in $\mathcal{Q}$. (See Figure 13.) The choice of d guarantees that for each n, $\gamma_n \subset \Omega$, and γ_n does not intersect K; for if it did, then it would belong to two adjacent squares in $\mathcal{Q}$, contradicting our choice of γ_n.

[5] These singularities are points where the function is not holomorphic, and are "poles", as defined in the next chapter.

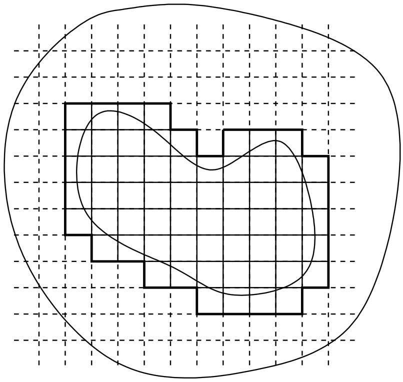

Figure 13. The union of the γ_n's is in bold-face

Since for any $z \in K$ that is not on the boundary of a square in $\mathcal{Q}$ there exists j so that $z \in Q_j$, Cauchy's theorem implies

$$\frac{1}{2\pi i}\int_{\partial Q_m} \frac{f(\zeta)}{\zeta - z}\,d\zeta = \begin{cases} f(z) & \text{if } m = j, \\ 0 & \text{if } m \neq j. \end{cases}$$

Thus, for all such z we have

$$f(z) = \sum_{m=1}^{M} \frac{1}{2\pi i}\int_{\partial Q_m} \frac{f(\zeta)}{\zeta - z}\,d\zeta.$$

However, if Q_m and $Q_{m'}$ are adjacent, the integral over their common side is taken once in each direction, and these cancel. This establishes (15) when z is in K and not on the boundary of a square in $\mathcal{Q}$. Since $\gamma_n \subset K^c$, continuity guarantees that this relation continues to hold for all $z \in K$, as was to be shown.

The first part of Theorem 5.7 is therefore a consequence of the next lemma.

Lemma 5.9 *For any line segment γ entirely contained in $\Omega - K$, there exists a sequence of rational functions with singularities on γ that approximate the integral $\int_\gamma f(\zeta)/(\zeta - z)\,d\zeta$ uniformly on K.*

Proof. If $\gamma(t) : [0,1] \to \mathbb{C}$ is a parametrization for γ, then

$$\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta = \int_0^1 \frac{f(\gamma(t))}{\gamma(t) - z}\gamma'(t)\,dt.$$

Since γ does not intersect K, the integrand $F(z,t)$ in this last integral is jointly continuous on $K \times [0,1]$, and since K is compact, given $\epsilon > 0$, there exists $\delta > 0$ such that

$$\sup_{z\in K} |F(z,t_1) - F(z,t_2)| < \epsilon \quad \text{whenever } |t_1 - t_2| < \delta.$$

Arguing as in the proof of Theorem 5.4, we see that the Riemann sums of the integral $\int_0^1 F(z,t)\,dt$ approximate it uniformly on K. Since each of these Riemann sums is a rational function with singularities on γ, the lemma is proved.

Finally, the process of pushing the poles to infinity is accomplished by using the fact that K^c is connected. Since any rational function whose only singularity is at the point z_0 is a polynomial in $1/(z - z_0)$, it suffices to establish the next lemma to complete the proof of Theorem 5.7.

Lemma 5.10 *If K^c is connected and $z_0 \notin K$, then the function $1/(z - z_0)$ can be approximated uniformly on K by polynomials.*

Proof. First, we choose a point z_1 that is outside a large open disc D centered at the origin and which contains K. Then

$$\frac{1}{z - z_1} = -\frac{1}{z_1}\frac{1}{1 - z/z_1} = \sum_{n=1}^{\infty} -\frac{z^n}{z_1^{n+1}},$$

where the series converges uniformly for $z \in K$. The partial sums of this series are polynomials that provide a uniform approximation to $1/(z - z_1)$ on K. In particular, this implies that any power $1/(z - z_1)^k$ can also be approximated uniformly on K by polynomials.

It now suffices to prove that $1/(z - z_0)$ can be approximated uniformly on K by polynomials in $1/(z - z_1)$. To do so, we use the fact that K^c is connected to travel from z_0 to the point z_1. Let γ be a curve in K^c that is parametrized by $\gamma(t)$ on $[0,1]$, and such that $\gamma(0) = z_0$ and $\gamma(1) = z_1$. If we let $\rho = \frac{1}{2}d(K,\gamma)$, then $\rho > 0$ since γ and K are compact. We then choose a sequence of points $\{w_1, \ldots, w_\ell\}$ on γ such that $w_0 = z_0$, $w_\ell = z_1$, and $|w_j - w_{j+1}| < \rho$ for all $0 \le j < \ell$.

We claim that if w is a point on γ, and w' any other point with $|w - w'| < \rho$, then $1/(z - w)$ can be approximated uniformly on K by polynomials in $1/(z - w')$. To see this, note that

$$\begin{aligned}\frac{1}{z - w} &= \frac{1}{z - w'}\frac{1}{1 - \frac{w - w'}{z - w'}} \\ &= \sum_{n=0}^{\infty} \frac{(w - w')^n}{(z - w')^{n+1}},\end{aligned}$$

and since the sum converges uniformly for $z \in K$, the approximation by partial sums proves our claim.

This result allows us to travel from z_0 to z_1 through the finite sequence $\{w_j\}$ to find that $1/(z - z_0)$ can be approximated uniformly on K by polynomials in $1/(z - z_1)$. This concludes the proof of the lemma, and also that of the theorem.

6 Exercises

1. Prove that

$$\int_0^\infty \sin(x^2)\,dx = \int_0^\infty \cos(x^2)\,dx = \frac{\sqrt{2\pi}}{4}.$$

These are the **Fresnel integrals**. Here, $\int_0^\infty$ is interpreted as $\lim_{R\to\infty} \int_0^R$.

[Hint: Integrate the function e^{-z^2} over the path in Figure 14. Recall that $\int_{-\infty}^\infty e^{-x^2}\,dx = \sqrt{\pi}$.]

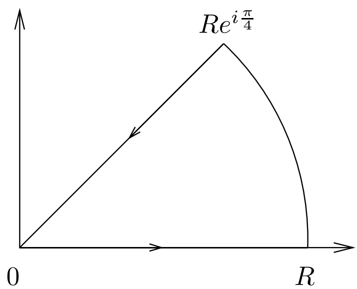

Figure 14. The contour in Exercise 1

2. Show that $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$

[Hint: The integral equals $\frac{1}{2i}\int_{-\infty}^\infty \frac{e^{ix}-1}{x}\,dx$. Use the indented semicircle.]

3. Evaluate the integrals

$$\int_0^\infty e^{-ax}\cos bx\,dx \quad \text{and} \quad \int_0^\infty e^{-ax}\sin bx\,dx\,, \qquad a > 0$$

by integrating e^{-Az}, $A = \sqrt{a^2 + b^2}$, over an appropriate sector with angle ω, with $\cos\omega = a/A$.

Corollary 2.3 *The only automorphisms of the unit disc that fix the origin are the rotations.*

Note that by the use of the mappings ψ_α, we can see that the group of automorphisms of the disc acts **transitively**, in the sense that given any pair of points α and β in the disc, there is an automorphism ψ mapping α to β. One such ψ is given by $\psi = \psi_\beta \circ \psi_\alpha$.

The explicit formulas for the automorphisms of $\mathbb{D}$ give a good description of the group $\mathrm{Aut}(\mathbb{D})$. In fact, this group of automorphisms is "almost" isomorphic to a group of 2×2 matrices with complex entries often denoted by $\mathrm{SU}(1,1)$. This group consists of all 2×2 matrices that preserve the hermitian form on $\mathbb{C}^2 \times \mathbb{C}^2$ defined by

$$\langle Z, W \rangle = z_1 \overline{w}_1 - z_2 \overline{w}_2,$$

where $Z = (z_1, z_2)$ and $W = (w_1, w_2)$. For more information about this subject, we refer the reader to Problem 4.

2.2 Automorphisms of the upper half-plane

Our knowledge of the automorphisms of $\mathbb{D}$ together with the conformal map $F : \mathbb{H} \to \mathbb{D}$ found in Section 1.1 allow us to determine the group of automorphisms of $\mathbb{H}$ which we denote by $\mathrm{Aut}(\mathbb{H})$.

Consider the map

$$\Gamma : \mathrm{Aut}(\mathbb{D}) \to \mathrm{Aut}(\mathbb{H})$$

given by "conjugation by F":

$$\Gamma(\varphi) = F^{-1} \circ \varphi \circ F.$$

It is clear that $\Gamma(\varphi)$ is an automorphism of $\mathbb{H}$ whenever φ is an automorphism of $\mathbb{D}$, and Γ is a bijection whose inverse is given by $\Gamma^{-1}(\psi) = F \circ \psi \circ F^{-1}$. In fact, we prove more, namely that Γ preserves the operations on the corresponding groups of automorphisms. Indeed, suppose that $\varphi_1, \varphi_2 \in \mathrm{Aut}(\mathbb{D})$. Since $F \circ F^{-1}$ is the identity on $\mathbb{D}$ we find that

$$\begin{aligned}
\Gamma(\varphi_1 \circ \varphi_2) &= F^{-1} \circ \varphi_1 \circ \varphi_2 \circ F \\
&= F^{-1} \circ \varphi_1 \circ F \circ F^{-1} \circ \varphi_2 \circ F \\
&= \Gamma(\varphi_1) \circ \Gamma(\varphi_2).
\end{aligned}$$

The conclusion is that the two groups $\mathrm{Aut}(\mathbb{D})$ and $\mathrm{Aut}(\mathbb{H})$ are the same, since Γ defines an isomorphism between them. We are still left with the task of giving a description of elements of $\text{Aut}(\mathbb{H})$. A series of calculations, which consist of pulling back the automorphisms of the disc to the upper half-plane via F, can be used to verify that $\text{Aut}(\mathbb{H})$ consists of all maps

$$z \mapsto \frac{az+b}{cz+d},$$

where a, b, c, and d are real numbers with $ad - bc = 1$. Again, a matrix group is lurking in the background. Let $\text{SL}_2(\mathbb{R})$ denote the group of all 2×2 matrices with real entries and determinant 1, namely

$$\text{SL}_2(\mathbb{R}) = \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \quad a, b, c, d \in \mathbb{R} \text{ and } \det(M) = ad - bc{=}1 \right\}.$$

This group is called the **special linear group**.

Given a matrix $M \in \text{SL}_2(\mathbb{R})$ we define the mapping f_M by

$$f_M(z) = \frac{az+b}{cz+d}.$$

Theorem 2.4 *Every automorphism of* $\mathbb{H}$ *takes the form* f_M *for some* $M \in \text{SL}_2(\mathbb{R})$. *Conversely, every map of this form is an automorphism of* $\mathbb{H}$.

The proof consists of a sequence of steps. For brevity, we denote the group $\text{SL}_2(\mathbb{R})$ by $\mathcal{G}$.

Step 1. If $M \in \mathcal{G}$, then f_M maps $\mathbb{H}$ to itself. This is clear from the observation that

$$(4) \quad \text{Im}(f_M(z)) = \frac{(ad-bc)\text{Im}(z)}{|cz+d|^2} = \frac{\text{Im}(z)}{|cz+d|^2} > 0 \quad \text{ whenever } z \in \mathbb{H}.$$

Step 2. If M and M' are two matrices in $\mathcal{G}$, then $f_M \circ f_{M'} = f_{MM'}$. This follows from a straightforward calculation, which we omit. As a consequence, we can prove the first half of the theorem. Each f_M is an automorphism because it has a holomorphic inverse $(f_M)^{-1}$, which is simply $f_{M^{-1}}$. Indeed, if I is the identity matrix, then

$$(f_M \circ f_{M^{-1}})(z) = f_{MM^{-1}}(z) = f_I(z) = z.$$

Step 3. Given any two points z and w in $\mathbb{H}$, there exists $M \in \mathcal{G}$ such that $f_M(z) = w$, and therefore $\mathcal{G}$ acts transitively on $\mathbb{H}$. To prove this, it suffices to show that we can map any $z \in \mathbb{H}$ to i. Setting $d = 0$ in equation (4) above gives

$$\operatorname{Im}(f_M(z)) = \frac{\operatorname{Im}(z)}{|cz|^2}$$

and we may choose a real number c so that $\operatorname{Im}(f_M(z)) = 1$. Next we choose the matrix

$$M_1 = \begin{pmatrix} 0 & -c^{-1} \\ c & 0 \end{pmatrix}$$

so that $f_{M_1}(z)$ has imaginary part equal to 1. Then we translate by a matrix of the form

$$M_2 = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \qquad \text{with } b \in \mathbb{R},$$

to bring $f_{M_1}(z)$ to i. Finally, the map f_M with $M = M_2 M_1$ takes z to i.

Step 4. If θ is real, then the matrix

$$M_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

belongs to $\mathcal{G}$, and if $F : \mathbb{H} \to \mathbb{D}$ denotes the standard conformal map, then $F \circ f_{M_\theta} \circ F^{-1}$ corresponds to the rotation of angle -2θ in the disc. This follows from the fact that $F \circ f_{M_\theta} = e^{-2i\theta} F(z)$, which is easily verified.

Step 5. We can now complete the proof of the theorem. We suppose f is an automorphism of $\mathbb{H}$ with $f(\beta) = i$, and consider a matrix $N \in \mathcal{G}$ such that $f_N(i) = \beta$. Then $g = f \circ f_N$ satisfies $g(i) = i$, and therefore $F \circ g \circ F^{-1}$ is an automorphism of the disc that fixes the origin. So $F \circ g \circ F^{-1}$ is a rotation, and by Step 4 there exists $\theta \in \mathbb{R}$ such that

$$F \circ g \circ F^{-1} = F \circ f_{M_\theta} \circ F^{-1}.$$

Hence $g = f_{M_\theta}$, and we conclude that $f = f_{M_\theta N^{-1}}$ which is of the desired form.

A final observation is that the group $\operatorname{Aut}(\mathbb{H})$ is not quite isomorphic with $\mathrm{SL}_2(\mathbb{R})$. The reason for this is because the two matrices M and $-M$ give rise to the same function $f_M = f_{-M}$. Therefore, if we identify the two matrices M and $-M$, then we obtain a new group $\mathrm{PSL}_2(\mathbb{R})$ called the **projective special linear group**; this group is isomorphic with $\operatorname{Aut}(\mathbb{H})$.