Theorem 5.6 (Schwarz reflection principle) Suppose that \( f \) is a holomorphic function in \( \Omega^+ \) that extends continuously to \( I \) and such that \( f \) is real-valued on \( I \). Then there exists a function \( F \) holomorphic in all of \( \Omega \) such that \( F = f \) on \( \Omega^+ \).

Proof. The idea is simply to define \( F(z) \) for \( z \in \Omega^- \) by
\[
F(z) = \overline{f(\overline{z})}.
\]
To prove that \( F \) is holomorphic in \( \Omega^- \) we note that if \( z, z_0 \in \Omega^- \), then \( \overline{z}, \overline{z_0} \in \Omega^+ \) and hence, the power series expansion of \( f \) near \( \overline{z_0} \) gives
\[
f(\overline{z}) = \sum a_n (\overline{z} - \overline{z_0})^n.
\]
As a consequence we see that
\[
F(z) = \sum \overline{a_n} (z - z_0)^n
\]
and \( F \) is holomorphic in \( \Omega^- \). Since \( f \) is real valued on \( I \) we have \( \overline{f(x)} = f(x) \) whenever \( x \in I \) and hence \( F \) extends continuously up to \( I \). The proof is complete once we invoke the symmetry principle.

5.5 Runge’s approximation theorem

We know by Weierstrass’s theorem that any continuous function on a compact interval can be approximated uniformly by polynomials.\(^4\) With this result in mind, one may inquire about similar approximations in complex analysis. More precisely, we ask the following question: what conditions on a compact set \( K \subset \mathbb{C} \) guarantee that any function holomorphic in a neighborhood of this set can be approximated uniformly by polynomials on \( K \)?

An example of this is provided by power series expansions. We recall that if \( f \) is a holomorphic function in a disc \( D \), then it has a power series expansion \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) that converges uniformly on every compact set \( K \subset D \). By taking partial sums of this series, we conclude that \( f \) can be approximated uniformly by polynomials on any compact subset of \( D \).

In general, however, some condition on \( K \) must be imposed, as we see by considering the function \( f(z) = 1/z \) on the unit circle \( K = C \). Indeed, recall that \( \int_C f(z) \, dz = 2\pi i \), and if \( p \) is any polynomial, then Cauchy’s theorem implies \( \int_C p(z) \, dz = 0 \), and this quickly leads to a contradiction.

\(^4\)A proof may be found in Section 1.8, Chapter 5, of Book 1.
A restriction on $K$ that guarantees the approximation pertains to the topology of its complement: $K^c$ must be connected. In fact, a slight modification of the above example when $f(z) = 1/z$ proves that this condition on $K$ is also necessary; see Problem 4.

Conversely, uniform approximations exist when $K^c$ is connected, and this result follows from a theorem of Runge which states that for any $K$ a uniform approximation exists by rational functions with “singularities” in the complement of $K$.\(^5\) This result is remarkable since rational functions are globally defined, while $f$ is given only in a neighborhood of $K$. In particular, $f$ could be defined independently on different components of $K$, making the conclusion of the theorem even more striking.

**Theorem 5.7** Any function holomorphic in a neighborhood of a compact set $K$ can be approximated uniformly on $K$ by rational functions whose singularities are in $K^c$.

If $K^c$ is connected, any function holomorphic in a neighborhood of $K$ can be approximated uniformly on $K$ by polynomials.

We shall see how the second part of the theorem follows from the first: when $K^c$ is connected, one can “push” the singularities to infinity thereby transforming the rational functions into polynomials.

The key to the theorem lies in an integral representation formula that is a simple consequence of the Cauchy integral formula applied to a square.

**Lemma 5.8** Suppose $f$ is holomorphic in an open set $\Omega$, and $K \subset \Omega$ is compact. Then, there exists finitely many segments $\gamma_1, \ldots, \gamma_N$ in $\Omega - K$ such that

\[
(15) \quad f(z) = \sum_{n=1}^{N} \frac{1}{2\pi i} \int_{\gamma_n} \frac{f(\zeta)}{\zeta - z} \, d\zeta \quad \text{for all } z \in K.
\]

**Proof.** Let $d = c \cdot d(K, \Omega^c)$, where $c$ is any constant $< 1/\sqrt{2}$, and consider a grid formed by (solid) squares with sides parallel to the axis and of length $d$.

We let $Q = \{Q_1, \ldots, Q_M\}$ denote the finite collection of squares in this grid that intersect $K$, with the boundary of each square given the positive orientation. (We denote by $\partial Q_m$ the boundary of the square $Q_m$.) Finally, we let $\gamma_1, \ldots, \gamma_N$ denote the sides of squares in $Q$ that do not belong to two adjacent squares in $Q$. (See Figure 13.) The choice of $d$ guarantees that for each $n$, $\gamma_n \subset \Omega$, and $\gamma_n$ does not intersect $K$; for if it did, then it would belong to two adjacent squares in $Q$, contradicting our choice of $\gamma_n$.

\(^5\)These singularities are points where the function is not holomorphic, and are “poles”, as defined in the next chapter.
Since for any \( z \in K \) that is not on the boundary of a square in \( Q \) there exists \( j \) so that \( z \in Q_j \), Cauchy’s theorem implies
\[
\frac{1}{2\pi i} \int_{\partial Q_m} \frac{f(\zeta)}{\zeta - z} d\zeta = \begin{cases} 
 f(z) & \text{if } m = j, \\
 0 & \text{if } m \neq j.
\end{cases}
\]
Thus, for all such \( z \) we have
\[
f(z) = \sum_{m=1}^{M} \frac{1}{2\pi i} \int_{\partial Q_m} \frac{f(\zeta)}{\zeta - z} d\zeta.
\]
However, if \( Q_m \) and \( Q_{m'} \) are adjacent, the integral over their common side is taken once in each direction, and these cancel. This establishes (15) when \( z \) is in \( K \) and not on the boundary of a square in \( Q \). Since \( \gamma_n \subset K^c \), continuity guarantees that this relation continues to hold for all \( z \in K \), as was to be shown.

The first part of Theorem 5.7 is therefore a consequence of the next lemma.

**Lemma 5.9** For any line segment \( \gamma \) entirely contained in \( \Omega - K \), there exists a sequence of rational functions with singularities on \( \gamma \) that approximate the integral \( \int_{\gamma} f(\zeta)/(\zeta - z) \, d\zeta \) uniformly on \( K \).

**Proof.** If \( \gamma(t) : [0, 1] \to \mathbb{C} \) is a parametrization for \( \gamma \), then
\[
\int_{\gamma} \frac{f(\zeta)}{\zeta - z} \, d\zeta = \int_{0}^{1} \frac{f(\gamma(t))}{\gamma(t) - z} \gamma'(t) \, dt.
\]
Since $\gamma$ does not intersect $K$, the integrand $F(z, t)$ in this last integral is jointly continuous on $K \times [0, 1]$, and since $K$ is compact, given $\epsilon > 0$, there exists $\delta > 0$ such that

$$\sup_{z \in K} |F(z, t_1) - F(z, t_2)| < \epsilon \quad \text{whenever } |t_1 - t_2| < \delta.$$ 

Arguing as in the proof of Theorem 5.4, we see that the Riemann sums of the integral $\int_0^1 F(z, t) \, dt$ approximate it uniformly on $K$. Since each of these Riemann sums is a rational function with singularities on $\gamma$, the lemma is proved.

Finally, the process of pushing the poles to infinity is accomplished by using the fact that $K^c$ is connected. Since any rational function whose only singularity is at the point $z_0$ is a polynomial in $1/(z - z_0)$, it suffices to establish the next lemma to complete the proof of Theorem 5.7.

**Lemma 5.10** If $K^c$ is connected and $z_0 \notin K$, then the function $1/(z - z_0)$ can be approximated uniformly on $K$ by polynomials.

**Proof.** First, we choose a point $z_1$ that is outside a large open disc $D$ centered at the origin and which contains $K$. Then

$$\frac{1}{z - z_1} = -\frac{1}{z_1} \frac{1}{1 - z/z_1} = \sum_{n=1}^{\infty} -\frac{z^n}{z_1^{n+1}},$$

where the series converges uniformly for $z \in K$. The partial sums of this series are polynomials that provide a uniform approximation to $1/(z - z_1)$ on $K$. In particular, this implies that any power $1/(z - z_1)^k$ can also be approximated uniformly on $K$ by polynomials.

It now suffices to prove that $1/(z - z_0)$ can be approximated uniformly on $K$ by polynomials in $1/(z - z_1)$. To do so, we use the fact that $K^c$ is connected to travel from $z_0$ to the point $z_1$. Let $\gamma$ be a curve in $K^c$ that is parametrized by $\gamma(t)$ on $[0, 1]$, and such that $\gamma(0) = z_0$ and $\gamma(1) = z_1$. If we let $\rho = \frac{1}{2}d(K, \gamma)$, then $\rho > 0$ since $\gamma$ and $K$ are compact. We then choose a sequence of points $\{w_1, \ldots, w_\ell\}$ on $\gamma$ such that $w_0 = z_0$, $w_\ell = z_1$, and $|w_j - w_{j+1}| < \rho$ for all $0 \leq j < \ell$.

We claim that if $w$ is a point on $\gamma$, and $w'$ any other point with $|w - w'| < \rho$, then $1/(z - w)$ can be approximated uniformly on $K$ by polynomials in $1/(z - w')$. To see this, note that

$$\frac{1}{z - w} = \frac{1}{z - w'} \frac{1}{1 - \frac{w - w'}{z - w'}} = \sum_{n=0}^{\infty} \frac{(w - w')^n}{(z - w')^{n+1}}.$$
and since the sum converges uniformly for \( z \in K \), the approximation by partial sums proves our claim.

This result allows us to travel from \( z_0 \) to \( z_1 \) through the finite sequence \( \{w_j\} \) to find that \( 1/(z - z_0) \) can be approximated uniformly on \( K \) by polynomials in \( 1/(z - z_1) \). This concludes the proof of the lemma, and also that of the theorem.

## 6 Exercises

1. Prove that

\[
\int_0^\infty \sin(x^2) \, dx = \int_0^\infty \cos(x^2) \, dx = \frac{\sqrt{2\pi}}{4}.
\]

These are the **Fresnel integrals**. Here, \( \int_0^\infty \) is interpreted as \( \lim_{R \to \infty} \int_0^R \).

[Hint: Integrate the function \( e^{-z^2} \) over the path in Figure 14. Recall that \( \int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi} \).

2. Show that

\[
\int_0^\infty \frac{\sin x}{x} \, dx = \frac{\pi}{2}.
\]

[Hint: The integral equals \( \frac{1}{2} \int_{-\infty}^{\infty} \frac{e^{ix} - 1}{x} \, dx \). Use the indented semicircle.]

3. Evaluate the integrals

\[
\int_0^\infty e^{-ax} \cos bx \, dx \quad \text{and} \quad \int_0^\infty e^{-ax} \sin bx \, dx, \quad a > 0
\]

by integrating \( e^{-Ax} \), \( A = \sqrt{a^2 + b^2} \), over an appropriate sector with angle \( \omega \), with \( \cos \omega = a/A \).
Corollary 2.3 The only automorphisms of the unit disc that fix the origin are the rotations.

Note that by the use of the mappings $\psi_\alpha$, we can see that the group of automorphisms of the disc acts transitively, in the sense that given any pair of points $\alpha$ and $\beta$ in the disc, there is an automorphism $\psi$ mapping $\alpha$ to $\beta$. One such $\psi$ is given by $\psi = \psi_\beta \circ \psi_\alpha$.

The explicit formulas for the automorphisms of $D$ give a good description of the group $\text{Aut}(D)$. In fact, this group of automorphisms is “almost” isomorphic to a group of $2 \times 2$ matrices with complex entries often denoted by $\text{SU}(1, 1)$. This group consists of all $2 \times 2$ matrices that preserve the hermitian form on $\mathbb{C}^2 \times \mathbb{C}^2$ defined by

$$\langle Z, W \rangle = z_1\overline{w}_1 - z_2\overline{w}_2,$$

where $Z = (z_1, z_2)$ and $W = (w_1, w_2)$. For more information about this subject, we refer the reader to Problem 4.

2.2 Automorphisms of the upper half-plane

Our knowledge of the automorphisms of $D$ together with the conformal map $F : \mathbb{H} \to D$ found in Section 1.1 allow us to determine the group of automorphisms of $\mathbb{H}$ which we denote by $\text{Aut}(\mathbb{H})$.

Consider the map

$$\Gamma : \text{Aut}(D) \to \text{Aut}(\mathbb{H})$$

given by “conjugation by $F$”:

$$\Gamma(\varphi) = F^{-1} \circ \varphi \circ F.$$

It is clear that $\Gamma(\varphi)$ is an automorphism of $\mathbb{H}$ whenever $\varphi$ is an automorphism of $D$, and $\Gamma$ is a bijection whose inverse is given by $\Gamma^{-1}(\psi) = F \circ \psi \circ F^{-1}$. In fact, we prove more, namely that $\Gamma$ preserves the operations on the corresponding groups of automorphisms. Indeed, suppose that $\varphi_1, \varphi_2 \in \text{Aut}(D)$. Since $F \circ F^{-1}$ is the identity on $D$ we find that

$$\Gamma(\varphi_1 \circ \varphi_2) = F^{-1} \circ \varphi_1 \circ \varphi_2 \circ F$$
$$= F^{-1} \circ \varphi_1 \circ F \circ F^{-1} \circ \varphi_2 \circ F$$
$$= \Gamma(\varphi_1) \circ \Gamma(\varphi_2).$$

The conclusion is that the two groups $\text{Aut}(D)$ and $\text{Aut}(\mathbb{H})$ are the same, since $\Gamma$ defines an isomorphism between them. We are still left with the
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task of giving a description of elements of Aut($\mathbb{H}$). A series of calculations, which consist of pulling back the automorphisms of the disc to the upper half-plane via $F$, can be used to verify that Aut($\mathbb{H}$) consists of all maps

$$z \mapsto \frac{az + b}{cz + d},$$

where $a, b, c,$ and $d$ are real numbers with $ad - bc = 1$. Again, a matrix group is lurking in the background. Let SL$_2(\mathbb{R})$ denote the group of all $2 \times 2$ matrices with real entries and determinant 1, namely

$$\text{SL}_2(\mathbb{R}) = \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \text{ and } \det(M) = ad - bc = 1 \right\}.$$

This group is called the special linear group.

Given a matrix $M \in \text{SL}_2(\mathbb{R})$ we define the mapping $f_M$ by

$$f_M(z) = \frac{az + b}{cz + d}.$$

**Theorem 2.4** Every automorphism of $\mathbb{H}$ takes the form $f_M$ for some $M \in \text{SL}_2(\mathbb{R})$. Conversely, every map of this form is an automorphism of $\mathbb{H}$.

The proof consists of a sequence of steps. For brevity, we denote the group $\text{SL}_2(\mathbb{R})$ by $G$.

**Step 1.** If $M \in G$, then $f_M$ maps $\mathbb{H}$ to itself. This is clear from the observation that

$$\text{Im}(f_M(z)) = \frac{(ad - bc)\text{Im}(z)}{|cz + d|^2} = \frac{\text{Im}(z)}{|cz + d|^2} > 0 \quad \text{whenever } z \in \mathbb{H}.$$

**Step 2.** If $M$ and $M'$ are two matrices in $G$, then $f_M \circ f_{M'} = f_{MM'}$. This follows from a straightforward calculation, which we omit. As a consequence, we can prove the first half of the theorem. Each $f_M$ is an automorphism because it has a holomorphic inverse $(f_M)^{-1}$, which is simply $f_{M^{-1}}$. Indeed, if $I$ is the identity matrix, then

$$(f_M \circ f_{M^{-1}})(z) = f_{MM^{-1}}(z) = f_I(z) = z.$$

**Step 3.** Given any two points $z$ and $w$ in $\mathbb{H}$, there exists $M \in G$ such that $f_M(z) = w$, and therefore $G$ acts transitively on $\mathbb{H}$. To prove this,
it suffices to show that we can map any $z \in \mathbb{H}$ to $i$. Setting $d = 0$ in equation (4) above gives

$$\text{Im}(f_M(z)) = \frac{\text{Im}(z)}{|c_2|^2}$$

and we may choose a real number $c$ so that $\text{Im}(f_M(z)) = 1$. Next we choose the matrix

$$M_1 = \begin{pmatrix} 0 & -c^{-1} \\ c & 0 \end{pmatrix}$$

so that $f_{M_1}(z)$ has imaginary part equal to 1. Then we translate by a matrix of the form

$$M_2 = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \quad \text{with } b \in \mathbb{R},$$

to bring $f_{M_1}(z)$ to $i$. Finally, the map $f_M$ with $M = M_2M_1$ takes $z$ to $i$.

**Step 4.** If $\theta$ is real, then the matrix

$$M_{\theta} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

belongs to $G$, and if $F : \mathbb{H} \rightarrow \mathbb{D}$ denotes the standard conformal map, then $F \circ f_{M_{\theta}} \circ F^{-1}$ corresponds to the rotation of angle $-2\theta$ in the disc. This follows from the fact that $F \circ f_{M_{\theta}} = e^{-2i\theta}F(z)$, which is easily verified.

**Step 5.** We can now complete the proof of the theorem. We suppose $f$ is an automorphism of $\mathbb{H}$ with $f(\beta) = i$, and consider a matrix $N \in G$ such that $f_N(i) = \beta$. Then $g = f \circ f_N$ satisfies $g(i) = i$, and therefore $F \circ g \circ F^{-1}$ is an automorphism of the disc that fixes the origin. So $F \circ g \circ F^{-1}$ is a rotation, and by Step 4 there exists $\theta \in \mathbb{R}$ such that

$$F \circ g \circ F^{-1} = F \circ f_{M_{\theta}} \circ F^{-1}.$$

Hence $g = f_{M_{\theta}}$, and we conclude that $f = f_{M_{\theta}N^{-1}}$ which is of the desired form.

A final observation is that the group $\text{Aut}(\mathbb{H})$ is not quite isomorphic with $\text{SL}_2(\mathbb{R})$. The reason for this is because the two matrices $M$ and $-M$ give rise to the same function $f_M = f_{-M}$. Therefore, if we identify the two matrices $M$ and $-M$, then we obtain a new group $\text{PSL}_2(\mathbb{R})$ called the **projective special linear group**; this group is isomorphic with $\text{Aut}(\mathbb{H})$. 