## 3. LOCAL PROPERTIES OF ANALYTIC FUNCTIONS

We have already proved that an analytic function has derivatives of all orders. In this section we will make a closer study of the local properties. It will include a classification of the *isolated singularities* of analytic functions.

***3.1. Removable Singularities. Taylor's Theorem.*** In Theorem 3 we introduced a weaker condition which could be substituted for analyticity at a finite number of points without affecting the end result. We showed moreover, in Theorem 5, that Cauchy's theorem in a circular disk remains true under these weaker conditions. This was an essential point in our derivation of Cauchy's integral formula, for we were required to apply Cauchy's theorem to a function of the form $(f(z) - f(a))/(z - a)$.

Finally, it was pointed out that Cauchy's integral formula remains valid in the presence of a finite number of exceptional points, all satisfying the fundamental condition of Theorem 3, provided that none of them coincides with $a$. This remark is more important than it may seem on the surface. Indeed, Cauchy's formula provides us with a representation of $f(z)$ through an integral which in its dependence on $z$ has the same character at the exceptional points as everywhere else. It follows that the exceptional points are such only by lack of information, and not by their intrinsic nature. Points with this character are called *removable singularities.* We shall prove the following precise theorem:

**Theorem 7.** *Suppose that $f(z)$ is analytic in the region $\Omega'$ obtained by omitting a point $a$ from a region $\Omega$. A necessary and sufficient condition that there exist an analytic function in $\Omega$ which coincides with $f(z)$ in $\Omega'$ is that* $\lim_{z \to a} (z - a)f(z) = 0$. *The extended function is uniquely determined.*

The necessity and the uniqueness are trivial since the extended function must be continuous at $a$. To prove the sufficiency we draw a circle $C$ about $a$ so that $C$ and its inside are contained in $\Omega$. Cauchy's formula is valid, and we can write

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)\, d\zeta}{\zeta - z}$$

for all $z \neq a$ inside of $C$. But the integral in the right-hand member represents an analytic function of $z$ throughout the inside of $C$. Consequently, the function which is equal to $f(z)$ for $z \neq a$ and which has the value

$$\frac{1}{2\pi i} \int_C \frac{f(\zeta)\, d\zeta}{\zeta - a} \tag{27}$$

for $z = a$ is analytic in $\Omega$. It is natural to denote the extended function by $f(z)$ and the value (27) by $f(a)$.

We apply this result to the function

$$F(z) = \frac{f(z) - f(a)}{z - a}$$

used in the proof of Cauchy's formula. It is not defined for $z = a$, but it satisfies the condition $\lim_{z \to a} (z - a)F(z) = 0$. The limit of $F(z)$ as $z$ tends to $a$ is $f'(a)$. Hence there exists an analytic function which is equal to $F(z)$ for $z \neq a$ and equal to $f'(a)$ for $z = a$. Let us denote this function by $f_1(z)$. Repeating the process we can define an analytic function $f_2(z)$ which equals $(f_1(z) - f_1(a))/(z - a)$ for $z \neq a$ and $f_1'(a)$ for $z = a$, and so on.

The recursive scheme by which $f_n(z)$ is defined can be written in the form

$$\begin{aligned} f(z) &= f(a) + (z - a)f_1(z) \\ f_1(z) &= f_1(a) + (z - a)f_2(z) \\ &\cdots\cdots\cdots\cdots \\ f_{n-1}(z) &= f_{n-1}(a) + (z - a)f_n(z). \end{aligned}$$

From these equations which are trivially valid also for $z = a$ we obtain

$$f(z) = f(a) + (z - a)f_1(a) + (z - a)^2 f_2(a) + \cdots + (z - a)^{n-1} f_{n-1}(a) + (z - a)^n f_n(z).$$

Differentiating $n$ times and setting $z = a$ we find

$$f^{(n)}(a) = n! f_n(a).$$

This determines the coefficients $f_n(a)$, and we obtain the following form of *Taylor's theorem:*

**Theorem 8.** *If $f(z)$ is analytic in a region $\Omega$, containing $a$, it is possible to write*

$$(28) \quad f(z) = f(a) + \frac{f'(a)}{1!}(z - a) + \frac{f''(a)}{2!}(z - a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n - 1)!}(z - a)^{n-1} + f_n(z)(z - a)^n,$$

*where $f_n(z)$ is analytic in $\Omega$.*

This finite development must be well distinguished from the infinite *Taylor series* which we will study later. It is, however, the finite development (28) which is the most useful for the study of the local properties of $f(z)$. Its usefulness is enhanced by the fact that $f_n(z)$ has a simple explicit expression as a line integral.

Using the same circle $C$ as before we have first

$$f_n(z) = \frac{1}{2\pi i} \int_C \frac{f_n(\zeta)\, d\zeta}{\zeta - z}.$$

For $f_n(\zeta)$ we substitute the expression obtained from (28). There will be one main term containing $f(\zeta)$. The remaining terms are, except for constant factors, of the form

$$F_\nu(a) = \int_C \frac{d\zeta}{(\zeta - a)^\nu(\zeta - z)}, \qquad \nu \geqq 1.$$

But

$$F_1(a) = \frac{1}{z - a} \int_C \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - a}\right) d\zeta = 0,$$

identically for all $a$ inside of $C$. By Lemma 3 we have $F_{\nu+1}(a) = F_1^{(\nu)}(a)/\nu!$ and thus $F_\nu(a) = 0$ for all $\nu \geqq 1$. Hence the expression for $f_n(z)$ reduces to

(29) $$f_n(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)\, d\zeta}{(\zeta - a)^n(\zeta - z)}.$$

The representation is valid inside of $C$.

***3.2. Zeros and Poles.*** If $f(a)$ and all derivatives $f^{(\nu)}(a)$ vanish, we can write by (28)

(30) $$f(z) = f_n(z)(z - a)^n$$

for any $n$. An estimate for $f_n(z)$ can be obtained by (29). The disk with the circumference $C$ has to be contained in the region $\Omega$ in which $f(z)$ is defined and analytic. The absolute value $|f(z)|$ has a maximum $M$ on $C$; if the radius of $C$ is denoted by $R$, we find

$$|f_n(z)| \leqq \frac{M}{R^{n-1}(R - |z - a|)}$$

for $|z - a| < R$. By (30) we have thus

$$|f(z)| \leqq \left(\frac{|z - a|}{R}\right)^n \cdot \frac{MR}{R - |z - a|}.$$

But $(|z - a|/R)^n \to 0$ for $n \to \infty$, since $|z - a| < R$. Hence $f(z) = 0$ inside of $C$.

We show now that $f(z)$ is identically zero in all of $\Omega$. Let $E_1$ be the set on which $f(z)$ and all derivatives vanish and $E_2$ the set on which the function or one of the derivatives is different from zero. $E_1$ is open by the above reasoning, and $E_2$ is open because the function and all derivatives are continuous. Therefore either $E_1$ or $E_2$ must be empty. If $E_2$ is empty, the function is identically zero. If $E_1$ is empty, $f(z)$ can never vanish together with all its derivatives.

Assume that $f(z)$ is not identically zero. Then, if $f(a) = 0$, there exists a first derivative $f^{(h)}(a)$ which is different from zero. We say then that $a$ is a *zero of order h*, and the result that we have just proved expresses that there are no zeros of infinite order. In this respect an analytic function has the same local behavior as a polynomial, and just as in the case of polynomials we find that it is possible to write $f(z) = (z - a)^h f_h(z)$ where $f_h(z)$ is analytic and $f_h(a) \neq 0$.

In the same situation, since $f_h(z)$ is continuous, $f_h(z) \neq 0$ in a neighborhood of $a$ and $z = a$ is the only zero of $f(z)$ in this neighborhood. In other words, the zeros of an analytic function which does not vanish identically are *isolated.* This property can also be formulated as a uniqueness theorem: *If $f(z)$ and $g(z)$ are analytic in $\Omega$, and if $f(z) = g(z)$ on a set which has an accumulation point in $\Omega$, then $f(z)$ is identically equal to $g(z)$.* The conclusion follows by consideration of the difference $f(z) - g(z)$.

Particular instances of this result which deserve to be quoted are the following: If $f(z)$ is identically zero in a subregion of $\Omega$, then it is identically zero in $\Omega$, and the same is true if $f(z)$ vanishes on an arc which does not reduce to a point. We can also say that an analytic function is uniquely determined by its values on any set with an accumulation point in the region of analyticity. This does not mean that we know of any way in which the values of the function can be computed.

We consider now a function $f(z)$ which is analytic in a neighborhood of $a$, except perhaps at $a$ itself. In other words, $f(z)$ shall be analytic in a region $0 < |z - a| < \delta$. The point $a$ is called an *isolated singularity* of $f(z)$. We have already treated the case of a removable singularity. Since we can then define $f(a)$ so that $f(z)$ becomes analytic in the disk $|z - a| < \delta$, it needs no further consideration.†

If $\lim_{z \to a} f(z) = \infty$, the point $a$ is said to be a *pole* of $f(z)$, and we set $f(a) = \infty$. There exists a $\delta' \leqq \delta$ such that $f(z) \neq 0$ for $0 < |z - a| < \delta'$. In this region the function $g(z) = 1/f(z)$ is defined and analytic. But the singularity of $g(z)$ at $a$ is removable, and $g(z)$ has an analytic exten-

† If $a$ is a removable singularity, $f(z)$ is frequently said to be *regular* at $a$; this term is sometimes used as a synonym for analytic.

sion with $g(a) = 0$. Since $g(z)$ does not vanish identically, the zero at $a$ has a finite order, and we can write $g(z) = (z - a)^h g_h(z)$ with $g_h(a) \neq 0$. The number $h$ is the *order* of the pole, and $f(z)$ has the representation $f(z) = (z - a)^{-h} f_h(z)$ where $f_h(z) = 1/g_h(z)$ is analytic and different from zero in a neighborhood of $a$. The nature of a pole is thus exactly the same as in the case of a rational function.

A function $f(z)$ which is analytic in a region $\Omega$, except for poles, is said to be *meromorphic* in $\Omega$. More precisely, to every $a \in \Omega$ there shall exist a neighborhood $|z - a| < \delta$, contained in $\Omega$, such that either $f(z)$ is analytic in the whole neighborhood, or else $f(z)$ is analytic for $0 < |z - a| < \delta$, and the isolated singularity is a pole. Observe that the poles of a meromorphic function are isolated *by definition*. The quotient $f(z)/g(z)$ of two analytic functions in $\Omega$ is a meromorphic function in $\Omega$, provided that $g(z)$ is not identically zero. The only possible poles are the zeros of $g(z)$, but a common zero of $f(z)$ and $g(z)$ can also be a removable singularity. If this is the case, the value of the quotient must be determined by continuity. More generally, the sum, the product, and the quotient of two meromorphic functions are meromorphic. The case of an identically vanishing denominator must be excluded, unless we wish to consider the constant $\infty$ as a meromorphic function.

For a more detailed discussion of isolated singularities, we consider the conditions (1) $\lim_{z \to a} |z - a|^\alpha |f(z)| = 0$, (2) $\lim_{z \to a} |z - a|^\alpha |f(z)| = \infty$, for real values of $\alpha$. If (1) holds for a certain $\alpha$, then it holds for all larger $\alpha$, and hence for some integer $m$. Then $(z - a)^m f(z)$ has a removable singularity and vanishes for $z = a$. Either $f(z)$ is identically zero, in which case (1) holds for all $\alpha$, or $(z - a)^m f(z)$ has a zero of finite order $k$. In the latter case it follows at once that (1) holds for all $\alpha > h = m - k$, while (2) holds for all $\alpha < h$. Assume now that (2) holds for some $\alpha$; then it holds for all smaller $\alpha$, and hence for some integer $n$. The function $(z - a)^n f(z)$ has a pole of finite order $l$, and setting $h = n + l$ we find again that (1) holds for $\alpha > h$ and (2) for $\alpha < h$. The discussion shows that there are three possibilities: (i) condition (1) holds for all $\alpha$, and $f(z)$ vanishes identically; (ii) there exists an integer $h$ such that (1) holds for $\alpha > h$ and (2) for $\alpha < h$; (iii) neither (1) nor (2) holds for any $\alpha$.

Case (i) is uninteresting. In case (ii) $h$ may be called the *algebraic order* of $f(z)$ at $a$. It is positive in case of a pole, negative in case of a zero, and zero if $f(z)$ is analytic but $\neq 0$ at $a$. The remarkable thing is that the order is always an integer; there is no single-valued analytic function which tends to 0 or $\infty$ like a fractional power of $|z - a|$.

In the case of a pole of order $h$, let us apply Theorem 8 to the analytic function $(z - a)^h f(z)$. We obtain a development of the form

$$(z - a)^h f(z) = B_h + B_{h-1}(z - a) + \cdots + B_1(z - a)^{h-1} + \varphi(z)(z - a)^h$$

where $\varphi(z)$ is analytic at $z = a$. For $z \neq a$ we can divide by $(z - a)^h$ and find

$$f(z) = B_h(z - a)^{-h} + B_{h-1}(z - a)^{-h+1} + \cdots + B_1(z - a)^{-1} + \varphi(z).$$

The part of this development which precedes $\varphi(z)$ is called the *singular part* of $f(z)$ at $z = a$. A pole has thus not only an order, but also a well-defined singular part. The difference of two functions with the same singular part is analytic at $a$.

In case (iii) the point $a$ is an *essential isolated singularity.* In the neighborhood of an essential singularity $f(z)$ is at the same time unbounded and comes arbitrarily close to zero. As a characterization of the complicated behavior of a function in the neighborhood of an essential singularity, we prove the following classical theorem of Weierstrass:

**Theorem 9.** *An analytic function comes arbitrarily close to any complex value in every neighborhood of an essential singularity.*

If the assertion were not true, we could find a complex number $A$ and a $\delta > 0$ such that $|f(z) - A| > \delta$ in a neighborhood of $a$ (except for $z = a$). For any $\alpha < 0$ we have then $\lim_{z \to a} |z - a|^\alpha |f(z) - A| = \infty$. Hence $a$ would not be an essential singularity of $f(z) - A$. Accordingly, there exists a $\beta$ with $\lim_{z \to a} |z - a|^\beta |f(z) - A| = 0$, and we are free to choose $\beta > 0$. Since in that case $\lim_{z \to a} |z - a|^\beta |A| = 0$ it would follow that $\lim_{z \to a} |z - a|^\beta |f(z)| = 0$, and $a$ would not be an essential singularity of $f(z)$. The contradiction proves the theorem.

The notion of isolated singularity applies also to functions which are analytic in a neighborhood $|z| > R$ of $\infty$. Since $f(\infty)$ is not defined, we treat $\infty$ as an isolated singularity, and by convention it has the same character of removable singularity, pole, or essential singularity as the singularity of $g(z) = f(1/z)$ at $z = 0$. If the singularity is nonessential, $f(z)$ has an algebraic order $h$ such that $\lim_{z \to \infty} z^{-h} f(z)$ is neither zero nor infinity, and for a pole the singular part is a polynomial in $z$. If $\infty$ is an essential singularity, the function has the property expressed by Theorem 9 in every neighborhood of infinity.

## EXERCISES

**1.** If $f(z)$ and $g(z)$ have the algebraic orders $h$ and $k$ at $z = a$, show that $fg$ has the order $h + k$, $f/g$ the order $h - k$, and $f + g$ an order which does not exceed max $(h,k)$.