It is this formula which is usually referred to as *Cauchy's integral formula.* We must remember that it is valid only when $n(\gamma,z) = 1$, and that we have proved it only when $f(z)$ is analytic in a disk.

EXERCISES

1. Compute

$$\int_{|z|=1} \frac{e^z}{z}\,dz.$$

2. Compute

$$\int_{|z|=2} \frac{dz}{z^2+1}$$

by decomposition of the integrand in partial fractions.

3. Compute

$$\int_{|z|=\rho} \frac{|dz|}{|z-a|^2}$$

under the condition $|a| \neq \rho$. *Hint:* make use of the equations $z\bar{z} = \rho^2$ and

$$|dz| = -i\rho\,\frac{dz}{z}.$$

2.3. Higher Derivatives. The representation formula (22) gives us an ideal tool for the study of the local properties of analytic functions. In particular we can now show that an analytic function has derivatives of all orders, which are then also analytic.

We consider a function $f(z)$ which is analytic in an arbitrary region Ω. To a point $a \in \Omega$ we determine a δ-neighborhood Δ contained in Ω, and in Δ a circle C about a. Theorem 6 can be applied to $f(z)$ in Δ. Since $n(C,a) = 1$ we have $n(C,z) = 1$ for all points z inside of C. For such z we obtain by (22)

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)\,d\zeta}{\zeta - z}.$$

Provided that the integral can be differentiated under the sign of integration we find

$$f'(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)\,d\zeta}{(\zeta - z)^2} \tag{23}$$

and

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)\,d\zeta}{(\zeta - z)^{n+1}}. \tag{24}$$

If the differentiations can be justified, we shall have proved the existence of all derivatives at the points inside of C. Since every point in Ω lies inside of some such circle, the existence will be proved in the whole region Ω. At the same time we shall have obtained a convenient representation formula for the derivatives.

For the justification we could either refer to corresponding theorems in the real case, or we could prove a general theorem concerning line integrals whose integrand depends analytically on a parameter. Actually, we shall prove only the following lemma which is all we need in the present case:

Lemma 3. *Suppose that $\varphi(\zeta)$ is continuous on the arc γ. Then the function*

$$F_n(z) = \int_\gamma \frac{\varphi(\zeta)\,d\zeta}{(\zeta - z)^n}$$

is analytic in each of the regions determined by γ, and its derivative is $F'_n(z) = nF_{n+1}(z)$.

We prove first that $F_1(z)$ is continuous. Let z_0 be a point not on γ, and choose the neighborhood $|z - z_0| < \delta$ so that it does not meet γ. By restricting z to the smaller neighborhood $|z - z_0| < \delta/2$ we attain that $|\zeta - z| > \delta/2$ for all $\zeta \in \gamma$. From

$$F_1(z) - F_1(z_0) = (z - z_0) \int_\gamma \frac{\varphi(\zeta)\,d\zeta}{(\zeta - z)(\zeta - z_0)}$$

we obtain at once

$$|F_1(z) - F_1(z_0)| < |z - z_0| \cdot \frac{2}{\delta^2} \int_\gamma |\varphi|\,|d\zeta|,$$

and this inequality proves the continuity of $F_1(z)$ at z_0.

From this part of the lemma, applied to the function $\varphi(\zeta)/(\zeta - z_0)$, we conclude that the difference quotient

$$\frac{F_1(z) - F_1(z_0)}{z - z_0} = \int_\gamma \frac{\varphi(\zeta)\,d\zeta}{(\zeta - z)(\zeta - z_0)}$$

tends to the limit $F_2(z_0)$ as $z \to z_0$. Hence it is proved that $F'_1(z) = F_2(z)$.

The general case is proved by induction. Suppose we have shown that $F'_{n-1}(z) = (n - 1)F_n(z)$. From the identity

$$F_n(z) - F_n(z_0)$$
$$= \left[\int_\gamma \frac{\varphi\,d\zeta}{(\zeta - z)^{n-1}(\zeta - z_0)} - \int_\gamma \frac{\varphi\,d\zeta}{(\zeta - z_0)^n}\right] + (z - z_0) \int_\gamma \frac{\varphi\,d\zeta}{(\zeta - z)^n(\zeta - z_0)}$$

we can conclude that $F_n(z)$ is continuous. Indeed, by the induction hypothesis, applied to $\varphi(\zeta)/(\zeta - z_0)$, the first term tends to zero for $z \to z_0$, and in the second term the factor of $z - z_0$ is bounded in a neighborhood of z_0. Now, if we divide the identity by $z - z_0$ and let z tend to z_0, the quotient in the first term tends to a derivative which by the induction hypothesis equals $(n-1)F_{n+1}(z_0)$. The remaining factor in the second term is continuous, by what we have already proved, and has the limit $F_{n+1}(z_0)$. Hence $F'_n(z_0)$ exists and equals $nF_{n+1}(z_0)$.

It is clear that Lemma 3 is just what is needed in order to deduce (23) and (24) in a rigorous way. We have thus proved that an analytic function has derivatives of all orders which are analytic and can be represented by the formula (24).

Among the consequences of this result we like to single out two classical theorems. The first is known as *Morera's theorem*, and it can be stated as follows:

If $f(z)$ is defined and continuous in a region Ω, and if $\int_\gamma f\,dz = 0$ for all closed curves γ in Ω, then $f(z)$ is analytic in Ω.

The hypothesis implies, as we have already remarked in Sec. 1.3, that $f(z)$ is the derivative of an analytic function $F(z)$. We know now that $f(z)$ is then itself analytic.

A second classical result goes under the name of *Liouville's theorem:*

A function which is analytic and bounded in the whole plane must reduce to a constant.

For the proof we make use of a simple estimate derived from (24). Let the radius of C be r, and assume that $|f(\zeta)| \leqq M$ on C. If we apply (24) with $z = a$, we obtain at once

$$|f^{(n)}(a)| \leqq Mn!r^{-n}. \tag{25}$$

For Liouville's theorem we need only the case $n = 1$. The hypothesis means that $|f(\zeta)| \leqq M$ on all circles. Hence we can let r tend to ∞, and (25) leads to $f'(a) = 0$ for all a. We conclude that the function is constant.

Liouville's theorem leads to an almost trivial proof of the *fundamental theorem of algebra.* Suppose that $P(z)$ is a polynomial of degree > 0. If $P(z)$ were never zero, the function $1/P(z)$ would be analytic in the whole plane. We know that $P(z) \to \infty$ for $z \to \infty$, and therefore $1/P(z)$ tends to zero. This implies boundedness (the absolute value is continuous on the Riemann sphere and has thus a finite maximum), and by Liouville's theorem $1/P(z)$ would be constant. Since this is not so, the equation $P(z) = 0$ must have a root.

The inequality (25) is known as *Cauchy's estimate.* It shows above all that the successive derivatives of an analytic function cannot be arbitrary; there must always exist an M and an r so that (25) is fulfilled. In order to make the best use of the inequality it is important that r be judiciously chosen, the object being to minimize the function $M(r)r^{-n}$, where $M(r)$ is the maximum of $|f|$ on $|\zeta - a| = r$.

EXERCISES

1. Compute

$$\int_{|z|=1} e^z z^{-n}\,dz, \qquad \int_{|z|=2} z^n(1-z)^m\,dz, \qquad \int_{|z|=\rho} |z-a|^{-4}\,|dz|\ (|a| \neq \rho).$$

2. Prove that a function which is analytic in the whole plane and satisfies an inequality $|f(z)| < |z|^n$ for some n and all sufficiently large $|z|$ reduces to a polynomial.

3. If $f(z)$ is analytic and $|f(z)| \leqq M$ for $|z| \leqq R$, find an upper bound for $|f^{(n)}(z)|$ in $|z| \leqq \rho < R$.

4. If $f(z)$ is analytic for $|z| < 1$ and $|f(z)| \leqq 1/(1 - |z|)$, find the best estimate of $|f^{(n)}(0)|$ that Cauchy's inequality will yield.

5. Show that the successive derivatives of an analytic function at a point can never satisfy $|f^{(n)}(z)| > n!n^n$. Formulate a sharper theorem of the same kind.

***6.** A more general form of Lemma 3 reads as follows:

Let the function $\varphi(z,t)$ be continuous as a function of both variables when z lies in a region Ω and $\alpha \leqq t \leqq \beta$. Suppose further that $\varphi(z,t)$ is analytic as a function of $z \in \Omega$ for any fixed t. Then

$$F(z) = \int_\alpha^\beta \varphi(z,t)\,dt$$

is analytic in z and

$$F'(z) = \int_\alpha^\beta \frac{\partial\varphi(z,t)}{\partial z}\,dt. \tag{26}$$

To prove this represent $\varphi(z,t)$ as a Cauchy integral

$$\varphi(z,t) = \frac{1}{2\pi i}\int_C \frac{\varphi(\zeta,t)}{\zeta - z}\,d\zeta.$$

Fill in the necessary details to obtain

$$F(z) = \int_C \left(\frac{1}{2\pi i}\int_\alpha^\beta \varphi(\zeta,t)\,dt\right)\frac{d\zeta}{\zeta - z}$$

and use Lemma 3 to prove (26).