6. Find the interior, closure, and boundary for the set $\{z \in \mathbb{C} : 1 \le |z| < 2\}$ (no proof required).
7. Prove that $w \in \mathbb{C}$ is in the closure of a set $E \subset \mathbb{C}$ if and only if there is a sequence $\{z_n\} \subset E$ such that $\lim z_n = w$. Thus, a set E is closed if and only if it contains all limits of convergent sequences of points in E.
8. Does $\lim_{z\to 0} f(z)$ exist if $f(z) = \dfrac{|z - \overline{z}|}{|z|}$ with domain $\mathbb{C} \setminus \{0\}$? How about if the domain is restricted to be just $\mathbb{R} \setminus \{0\}$?
9. Prove that $\mathrm{Re}(z)$, $\mathrm{Im}(z)$, and $\overline{z}$ are continuous functions of z.
10. At which points of $\mathbb{C}$ is the function $(1 - z^4)^{-1}$ continuous.
11. Prove that $\arg_I$ is continuous except on its cut line.
12. Use the result of the preceding exercise to prove that a branch of the log function is continuous except on its cut line.
13. Use Theorem 2.1.13 to prove that if f and g are continuous functions with open domains U_f and U_g and if $g(U_g) \subset U_f$, then $f \circ g$ is continuous on U_g.
14. Prove that if f is a continuous function defined on an open subset U of $\mathbb{C}$, then sets of the form $\{z \in U : |f(z)| < r\}$ and $\{z \in U : \mathrm{Re}(f(z)) < r\}$ are open.
15. Use the result of the preceding exercise to come up with an open subset of $\mathbb{C}$ that has not been previously described in this text.
16. Prove that a function f with open domain U is continuous at a point $a \in U$ if and only if whenever $\{z_n\} \subset U$ is a sequence converging to a, the sequence $\{f(z_n)\}$ converges to $f(a)$.

2.2. The Complex Derivative

There is nothing surprising about the definition of the derivative of a function of a complex variable – it looks just like the definition of the derivative of a function of a real variable. What is surprising are the consequences of a function having a derivative in this sense.

Definition 2.2.1. Let f be a function defined on a neighborhood of $z \in \mathbb{C}$. If

$$\lim_{w\to z} \frac{f(w) - f(z)}{w - z}$$

exists, then we denote it by $f'(z)$ and we say f is differentiable at z with complex derivative $f'(z)$. If f is defined and differentiable at every point of an open set U, then we say that f is *analytic* on U.

Remark 2.2.2. When convenient, we will make the change of variables $\lambda = w - z$ and write the derivative in the form

$$f'(z) = \lim_{\lambda\to 0} \frac{f(z+\lambda) - f(z)}{\lambda}. \tag{2.2.1}$$

Clearly constant functions are differentiable and have complex derivative 0, since the difference quotient in Definition 2.2.1 is identically 0 for such a function.

The first hint that there is something fundamentally different about this notion of derivative is in the following example.

Example 2.2.3. Show that the function $f(z) = z$ is differentiable everywhere on $\mathbb{C}$ with derivative 1 and, hence, is analytic on $\mathbb{C}$, but the function $f(z) = \overline{z}$ is differentiable nowhere.

Solution: For $f(z) = z$, the difference quotient in (2.2.1) is

$$\frac{\lambda}{\lambda} = 1,$$

which clearly has limit 1 as $\lambda \to 0$ for every z. On the other hand, if $f(z) = \overline{z}$, then the difference quotient is

$$\frac{\overline{\lambda}}{\lambda} = \mathrm{e}^{-2i\theta},$$

if $\lambda = r\,\mathrm{e}^{i\theta}$ in polar form. The limit of this function as $\lambda \to 0$ clearly does not exist, since it has a different fixed value along each ray emanating from 0. This is true no matter what z is, and so $\overline{z}$ is nowhere differentiable.

What makes this example so surprising, at first, is that, as a function of the two real variables x and y, $\overline{z} = x - iy$ is of class $\mathcal{C}^\infty$ – meaning that its partial derivatives of all orders exist and are continuous – and yet, its complex derivative does not exist. Thus, existence of the complex derivative involves more than just smoothness of the function.

We will soon prove that a function which has a power series expansion that converges on an open disc is analytic on that disc. This would imply that the exponential function, for example, is analytic on all of $\mathbb{C}$. We do not have to wait, however, to prove this fact. There is an elementary proof that e^z is analytic on $\mathbb{C}$.

Example 2.2.4. Prove that e^z is an analytic function of z on the entire complex plane and show that it is its own derivative.

Solution: Given an arbitrary point $z \in \mathbb{C}$, we will show that e^z has derivative e^z at z. By the law of exponents

$$\frac{\mathrm{e}^{z+\lambda} - \mathrm{e}^z}{\lambda} = \mathrm{e}^z\,\frac{\mathrm{e}^\lambda - 1}{\lambda}.$$

Thus, to show that the derivative of e^z is e^z we need only show that

$$\lim_{\lambda \to 0} \frac{\mathrm{e}^\lambda - 1}{\lambda} = 1. \tag{2.2.2}$$

However, if $t = |\lambda|$, inspection of the power series for e^λ and e^t shows that

$$\left|\frac{\mathrm{e}^\lambda - 1}{\lambda} - 1\right| = \left|\frac{\mathrm{e}^\lambda - 1 - \lambda}{\lambda}\right| \le \frac{\mathrm{e}^t - 1 - t}{t}. \tag{2.2.3}$$

Now to show that the expression on the left has limit zero and, thus, verify (2.2.2), we simply apply L'Hôpital's rule to the expression on the right.

Elementary Properties of the Derivative. A simple result about derivatives of functions of a real variable that also holds in the context of complex derivatives is the following. The proof is elementary and is left to the exercises.

Theorem 2.2.5. *If the complex derivative f' of f exists at $a \in \mathbb{C}$, then f is continuous at a.*

The complex derivative has all of the familiar properties in relation to sums, products, and quotients of functions. The proofs of these are in no way different from the proofs of the corresponding results for functions of a real variable. In the following theorem, Part (a) is trivial and we leave Parts (b) and (c) to the exercises.

Theorem 2.2.6. *If f and g are functions of a complex variable which are differentiable at $z \in \mathbb{C}$, then*

(a) *$f + g$ is differentiable at z and $(f+g)'(z) = f'(z) + g'(z)$;*

(b) *fg is differentiable at z and $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$;*

(c) *if $g(z) \neq 0$, $1/g$ is differentiable at z and $(1/g)'(z) = -g'(z)/g^2(z)$.*

Parts (a) and (b) of this theorem and the fact that constant functions and the function z are analytic on $\mathbb{C}$ imply that every polynomial in z is analytic on $\mathbb{C}$. Of course, since $\overline{z}$ is not analytic, we cannot expect mixed polynomials that contain powers of both z and $\overline{z}$ to be analytic.

Parts (b) and (c) of the theorem imply that f/g is differentiable at z if f and g are and if $g(z) \neq 0$. They also imply the quotient rule

$$\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - g'(z)f(z)}{g^2(z)}.$$

The chain rule also holds for the complex derivative.

Theorem 2.2.7. *If g is differentiable at a and f is differentiable at $b = g(a)$, then $f \circ g$ is differentiable at a and*

$$(f \circ g)'(a) = f'(g(a))g'(a).$$

Proof. Let U be a neighborhood of b on which f is defined. We define a function $h(w)$ on U in the following way

$$h(w) = \begin{cases} \dfrac{f(w) - f(b)}{w - b}, & \text{if } w \neq b; \\ f'(b), & \text{if } w = b. \end{cases}$$

Then h is continuous at b, since

$$f'(b) = \lim_{w \to b} \frac{f(w) - f(b)}{w - b}.$$

Also,

$$\frac{f \circ g(z) - f \circ g(a)}{z - a} = h(g(z))\frac{g(z) - g(a)}{z - a} \tag{2.2.4}$$

for all z in the deleted neighborhood $V = g^{-1}(U) \setminus \{a\}$ of a. If we take the limit of both sides of (2.2.4) and use the fact that f and h are continuous at b and g is continuous at a, we conclude that $(f \circ g)'(a) = f'(g(a))g'(a)$, as required. □

Example 2.2.8. Suppose $p(z)$ is a polynomial in z. Where is the function $e^{p(z)}$ analytic and what is its derivative?

Solution: Since e^z and $p(z)$ both are differentiable everywhere, so is the composition $e^{p(z)}$, by Theorem 2.2.7, and the derivative is

$$\left(e^{p(z)}\right)' = p'(z)\, e^{p(z)} .$$

The Cauchy-Riemann Equations. Since a function f of a complex variable may be regarded as a complex-valued function on a subset of $\mathbb{R}^2$, we can write it in the form

$$f(x+iy) = u(x,y) + iv(x,y), \tag{2.2.5}$$

where u and v are the real and imaginary parts of f, regarded as functions defined on a subset of $\mathbb{R}^2$. It is natural to ask what the existence of a complex derivative for f implies about the functions u and v as functions of the two real variables x and y. It is easy to see that it implies the existence of the partial derivatives u_x, u_y, v_x and v_y. In fact, it implies much more as the following discussion will show.

Recall that a function g of two real variables is said to be *differentiable* at (x,y) if there are numbers A and B such that

$$g(x+h, y+k) - g(x,y) = Ah + Bk + \epsilon(h,k),$$

where $\epsilon(h,k)/|(h,k)| \to 0$ as $(h,k) \to (0,0)$. If g is differentiable at (x,y), then the numbers A and B are the partial derivatives g_x and g_y at (x,y).

Suppose f is a complex-valued function defined in a neighborhood of $z \in \mathbb{C}$. If $M = f'(z)$ exists, then we may write

$$f(z+\lambda) - f(z) = M\lambda + \epsilon(\lambda), \tag{2.2.6}$$

where $\epsilon(\lambda)/\lambda \to 0$ as $\lambda \to 0$. In fact, $\epsilon(\lambda)$ is given by

$$\epsilon(\lambda) = f(z+\lambda) - f(z) - M\lambda,$$

and so, the fact that $\epsilon(\lambda)/\lambda \to 0$ as $\lambda \to 0$ is equivalent to the statement that $f'(z)$ exists and is equal to M.

If we write f, M, z, λ, and ϵ in terms of their real and imaginary parts: $f = u + iv$, $M = C + iD$, $z = x + iy$, $\lambda = h + ik$, and $\epsilon = \rho + i\omega$, then (2.2.6) becomes

$$\begin{aligned} u(x+h, y+k) + iv(x+h, y+k) - u(x,y) - iv(x,y) \\ = (C+iD)(h+ik) + \rho(h,k) + i\omega(h,k). \end{aligned} \tag{2.2.7}$$

On equating real and imaginary parts, this leads to the two equations

$$\begin{aligned} u(x+h, y+k) - u(x,y) &= Ch - Dk + \rho(h,k), \\ v(x+h, y+k) - v(x,y) &= Dh + Ck + \omega(h,k). \end{aligned} \tag{2.2.8}$$

The condition that $\epsilon(\lambda)/\lambda \to 0$ as $\lambda \to 0$ implies that $\rho(h,k)/|(h,k)| \to 0$ and $\omega(h,k)/|(h,k)| \to 0$ (note that $|(h,k)| = \sqrt{h^2+k^2} = |\lambda|$). Thus, we can draw two conclusions from the existence of $f'(z)$: (1) u and v are differentiable at (x,y), and (2) the partial derivatives of u and v at (x,y) are given by

$$\begin{aligned} u_x(x,y) &= C, \quad & u_y(x,y) &= -D, \\ v_x(x,y) &= D, \quad & v_y(x,y) &= C. \end{aligned} \tag{2.2.9}$$

A surprising consequence of this is that if f' exists at $z = x + iy$, then

$$u_x = v_y, \qquad u_y = -v_x \tag{2.2.10}$$

at (x, y). Equations (2.2.10) are the *Cauchy-Riemann equations*. Equations (2.2.9) also show that if f' exists at z, then $f'(z) = C + iD = u_x + iv_x = -i(u_y + iv_y)$. If we set $f_x = u_x + iv_x$ and $f_y = u_y + iv_y$, then this can be written as $f' = f_x = -if_y$ wherever f' exists.

The above discussion shows that, at any point where f has a complex derivative, its real and imaginary parts are differentiable functions and satisfy the Cauchy-Riemann equations. The converse is also true: If the real and imaginary parts of f are differentiable and satisfy the Cauchy-Riemann equations at a point $z = x + iy$, then $f'(z)$ exists. The proof of this is a matter of working backwards through the above discussion, beginning with the assumption that u and v are differentiable at (x, y), with partial derivatives that satisfy $u_x = v_y = C$ and $u_y = -v_x = -D$. This leads to (2.2.8), which eventually leads back to the conclusion that $C + iD$ is the derivative of f at $z = x + iy$. We leave the details to the exercises. The result is the following theorem.

Theorem 2.2.9. *If $f = u + iv$ is a complex-valued function defined in a neighborhood of $z \in \mathbb{C}$, with real and imaginary parts u and v, then f has a complex derivative at z if and only if u and v are differentiable and satisfy the Cauchy-Riemann equations* (2.2.10) *at $z = x + iy$. In this case,*

$$f' = f_x = -if_y.$$

Example 2.2.10. We already know that e^z is analytic everywhere. However, give a different proof of this by showing e^z satisfies the Cauchy-Riemann equations.

Solution: With $z = x + iy$, we write $e^z = e^x(\cos y + i \sin y)$. The real and imaginary parts of e^z are $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$. Thus,

$$\begin{aligned} u_x(x, y) &= e^x \cos y = v_y, \quad \text{and} \\ u_y(x, y) &= -e^x \sin y = -v_x. \end{aligned}$$

Example 2.2.11. Use the Cauchy-Riemann equations to prove that, for each branch of the log function, $\log(z)$ is analytic everywhere except on its cut line and has derivative $1/z$.

Solution: We first prove that the principal branch of the log function is analytic on the right half-plane $H = \{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}$. For $z \in H$ we have $z = x + iy = r\, e^{i\theta}$ where

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \tan^{-1}(y/x).$$

Thus, the principal branch of log on H is

$$\log(x + iy) = (1/2)\ln(x^2 + y^2) + i \tan^{-1}(y/x).$$

Taking partial derivatives yields

$$
\begin{aligned}
\frac{\partial}{\partial x}(1/2)\ln(x^2+y^2) &= \frac{x}{x^2+y^2},\\
\frac{\partial}{\partial x}\tan^{-1}(y/x) &= \frac{-y/x^2}{1+(y/x)^2} = \frac{-y}{x^2+y^2},\\
\frac{\partial}{\partial y}(1/2)\ln(x^2+y^2) &= \frac{y}{x^2+y^2},\\
\frac{\partial}{\partial y}\tan^{-1}(y/x) &= \frac{1/x}{1+(y/x)^2} = \frac{x}{x^2+y^2}.
\end{aligned} \tag{2.2.11}
$$

Thus, the Cauchy-Riemann equations are satisfied by the principal branch of the log function on H. Furthermore

$$(\log z)' = \frac{\partial}{\partial x}\log(x+iy) = \frac{x-iy}{x^2+y^2} = \frac{1}{z}.$$

Now if z is any point not on the negative real axis and not in H, then we simply rotate z into H. That is, we choose $\alpha = \pm\pi/2$ such that $e^{i\alpha} z \in H$. Then

$$\log z = \log(e^{i\alpha} z) - i\alpha.$$

Since log has derivative $1/w$ at $w = e^{i\alpha} z$, it follows from the chain rule that log has derivative $e^{i\alpha}/(e^{i\alpha} z) = 1/z$ at z. Thus, the principal branch of the log function is analytic with derivative $1/z$ at any point z not on its cut line.

The analogous statement for other branches of the log function also follows from a rotation argument, as above. That is, each such function is just the principal branch of the log function composed with a rotation.

Harmonic Functions. In the next chapter, we will prove that analytic functions are $\mathcal{C}^\infty$– that is, they have continuous complex derivatives of all orders. This, in particular, implies that analytic functions have continuous partial derivatives of all orders with respect to x and y. Assuming this result for the moment, we have

Theorem 2.2.12. *The real and imaginary parts of an analytic function on U are harmonic functions on U, meaning they satisfy Laplace's equation*

$$u_{xx} + u_{yy} = 0.$$

Proof. If $f = u + iv$ is an analytic function, then u and v satisfy the Cauchy-Riemann equations and so

$$u_{xx} = (u_x)_x = (v_y)_x = (v_x)_y = (-u_y)_y = -u_{yy}.$$

This shows that the real part of f satisfies Laplace's equation. Since v is the real part of the analytic function $-if$, it follows that v is also harmonic. Thus, both real and imaginary parts of an analytic function are harmonic. □

If u and v are harmonic functions such that the function $f = u + iv$ is analytic, then we say u and v are *harmonic conjugates* of one another.

Example 2.2.13. Prove that $u(x, y) = e^x \cos y$ is a harmonic function on all of $\mathbb{R}^2$ and find a harmonic conjugate for it.

Solution: The function u is the real part of $f(z) = \mathrm{e}^z$ and is, therefore, harmonic by the previous theorem. The imaginary part of f is $v(x, y) = \mathrm{e}^x \sin y$, and so this function v is a harmonic conjugate of u.

Exercise Set 2.2

1. Fill in the details in Example 2.2.4 by verifying the inequality (2.2.3) and showing that the limit of the expression on the right is 0.
2. Prove Theorem 2.2.5.
3. Prove Part (b) of Theorem 2.2.6.
4. Prove Part (c) of Theorem 2.2.6.
5. Use induction and Theorem 2.2.6 to show that $(z^n)' = nz^{n-1}$ if n is a non-negative integer.
6. Find the derivative of $z^7 + 5z^4 - 2z^3 + z^2 - 1$. Which results from this section are used in this calculation?
7. Find the derivative of e^{z^3}.
8. If we use the principal branch of the log function, at which points of $\mathbb{C}$ does $\dfrac{\log z}{z}$ have a complex derivative? What is its derivative at these points?
9. Finish the proof of Theorem 2.2.9 by showing that if $f = u + iv$, u and v are differentiable at z, and u and v satisfy the Cauchy-Riemann equations at z, then $f'(z)$ exists.
10. Use the Cauchy-Riemann equations to verify that the function $f(z) = z^2$ is analytic everywhere.
11. Describe all real-valued functions which are analytic on $\mathbb{C}$.
12. Derive the Cauchy-Riemann equations in polar coordinates:
$$\begin{aligned} u_r &= r^{-1} v_\theta, \\ u_\theta &= -r v_r \end{aligned}$$
by using the change of variable formulas $x = r\cos\theta$, $y = r\sin\theta$ and the chain rule.
13. We showed in Example 2.2.11 that each branch of the log function is analytic on the complex plane with its cut line removed. Use the Cauchy-Riemann equations in polar form (previous problem) to give another proof of this fact.
14. Assuming each branch of the log function is analytic, use the chain rule to give another prove that each such function has derivative $1/z$.
15. Use the Cauchy-Riemann equations to prove that if f is analytic on an open set U, then the function g defined by $g(z) = \overline{f(\overline{z})}$ is analytic on the set $\{\overline{z} : z \in U\}$.
16. Verify that the function $\log|z|$ is harmonic on $\mathbb{C} \setminus \{0\}$ and find a harmonic conjugate for it on the set consisting of $\mathbb{C}$ with the non-positive real axis removed.

2.3. Contour Integrals

Integration plays a key role in this subject – specifically, integration along curves in $\mathbb{C}$. A *curve* or *contour* in the plane $\mathbb{C}$ is a continuous function γ from an interval on the line into $\mathbb{C}$. Such an object is sometimes called a *parameterized curve* and the interval I is called the parameter interval. We will be interested in a particular kind of curve, one whose parameter interval is a closed bounded interval which can be subdivided into finitely many subintervals, on each of which γ is continuously differentiable.

Smooth Curves. Let $I = [a, b]$ be a closed interval on the real line and let $\gamma : I \to \mathbb{C}$ be a complex-valued function on I. If $c \in I$, then the derivative $\gamma'(c)$ of γ at c is defined in the usual way:

$$\gamma'(c) = \lim_{t \to c} \frac{\gamma(t) - \gamma(c)}{t - c}. \tag{2.3.1}$$

Of course, γ is complex-valued and so this limit should be interpreted as the type of limit discussed in Section 2.1. It can be calculated by expressing γ in terms of its real and imaginary parts, that is, by writing $\gamma(t) = x(t) + iy(t)$, where $x(t)$ and $y(t)$ are real-valued functions on I. Then $\gamma'(t) = x'(t) + iy'(t)$ (Exercise 2.3.6).

What about the endpoints a and b of the interval I? Should we either not talk about the derivative at the endpoints or, perhaps, use one-sided derivatives defined in terms of one-sided limits (limit from the right at a and limit from the left at b)? Actually, there is no need to do anything special at a and b or to exclude them. If the domain of γ is $[a, b]$, then our domain dependent definition of limit takes care of the problem. If $c = a$, the limit as $t \to a$ in (2.3.1) only involves values of t to the right of a, since only those are in the domain of the difference quotient that appears in this limit. Similarly, if $c = b$, the limit as $t \to b$ involves only points to the left of b. Thus, the derivatives at a and b that our definition leads to are what in calculus would be called the right derivative at a and the left derivative at b.

The curve γ is *differentiable* at c if the limit defining $\gamma'(c)$ exists. It is *continuously differentiable* or *smooth* on I if it is differentiable at every point of I and if the derivative is a continuous function on I. In this case we will write $\gamma \in \mathcal{C}^1(I)$.

Definition 2.3.1. A curve $\gamma : [a, b] \to \mathbb{C}$ in $\mathbb{C}$ is called *piecewise smooth* if there is a partition $a = a_0 < a_1 < \cdots < a_n = b$ of $[a, b]$ such that the restriction of γ to $[a_{j-1}, a_j]$ is smooth for each $j = 1, \cdots, n$. A curve which is piecewise smooth will be called a *path*.

With appropriate choices of parameterization, familiar geometric objects in $\mathbb{C}$ can be described as the image of a path.

Example 2.3.2. Find a path γ that traces once around the circle of radius r, centered at 0, in the counterclockwise direction. Describe γ'.

Solution: The smooth path $\gamma(t) = r\,e^{it}$, $t \in [0, 2\pi]$ does the job. Its derivative may be obtained by writing it as $r(\cos t + i \sin t)$ and differentiating the real and imaginary parts. The result is $\gamma'(t) = r(-\sin t + i \cos t) = ir\,e^{it}$.

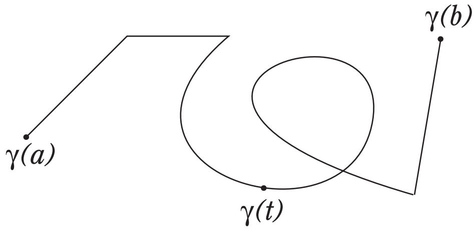

Figure 2.3.1. A Path in the Plane.

Example 2.3.3. Let z and w be two points in $\mathbb{C}$. Find a path which traces the straight line from z to w and find its derivative.

Solution: The path γ, with parameter interval $[0, 1]$, defined by

$$\gamma(t) = (1 - t)z + tw = z + t(w - z),$$

satisfies $\gamma(0) = z$ and $\gamma(1) = w$. It is a parametric form of a straight line in the plane, and its derivative is $\gamma'(t) = w - z$.

Example 2.3.4. Find a path that traces once around the square with vertices $0, 1, 1 + i, i$ in the counterclockwise direction. Find $\gamma'(t)$ on the subintervals where γ is smooth.

Solution: We choose $[0, 1]$ as the parameter interval and define a path γ as follows (see Figure 2.3.2):

$$\gamma(t) = \begin{cases} 4t, & \text{if } 0 \le t \le 1/4; \\ 1 + (4t - 1)i, & \text{if } 1/4 \le t \le 1/2; \\ 3 - 4t + i, & \text{if } 1/2 \le t \le 3/4; \\ (4 - 4t)i, & \text{if } 3/4 \le t \le 1. \end{cases}$$

This is continuous on $[0, 1]$ and smooth on each subinterval in the partition $0 < 1/4 < 1/2 < 3/4 < 1$. It traces each side of the square in succession, moving in the counterclockwise direction. On the first interval, γ' is the constant 4, on the second it is $4i$, on the third it is -4, and on the fourth it is $-4i$.

Riemann Integral of Complex-Valued Functions. The integral of a function along a path will be defined in terms of the Riemann integral on an interval. This is the familiar Riemann integral from calculus, except that the functions being integrated will be complex-valued. This difference requires a few comments.

If $f(t) = g(t) + ih(t)$ is a complex-valued function on an interval $[a, b]$, where g and h are real-valued, then we will say that f is *Riemann integrable* on $[a, b]$ if both g and h are Riemann integrable on $[a, b]$ as real-valued functions. We then define the integral of f on $[a, b]$ by

$$\int_a^b f(t)\,dt = \int_a^b g(t)\,dt + i\int_a^b h(t)\,dt. \tag{2.3.2}$$

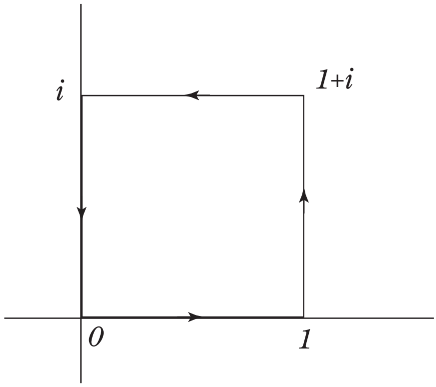

Figure 2.3.2. The Path of Example 2.3.4.

This Riemann integral for complex-valued functions has the properties one would expect given knowledge of the Riemann integral for real-valued functions. The next three theorems cover some of these properties.

Theorem 2.3.5. *Let f_1 and f_2 be Riemann integrable functions on $[a,b]$ and α and β complex numbers. Then, $\alpha f_1 + \beta f_2$ is integrable on $[a,b]$, and*

$$\int_a^b (\alpha f_1(t) + \beta f_2(t))\,dt = \alpha \int_a^b f_1(t)\,dt + \beta \int_a^b f_2(t)\,dt.$$

Proof. That this is true if the constants α and β are real follows directly from expressing f_1 and f_2 in terms of their real and imaginary parts. Thus, to prove the theorem we just need to show that $\int_a^b if(t)\,dt = i\int_a^b f(t)\,dt$ if $f = g + ih$ is an integrable function on $[a,b]$. However,

$$\begin{aligned}\int_a^b i(g(t)+ih(t))\,dt &= \int_a^b (-h(t)+ig(t))\,dt \\ &= -\int_a^b h(t)\,dt + i\int_a^b g(t)\,dt = i\left(\int_a^b (g(t)+ih(t))\,dt\right).\end{aligned}$$

This completes the proof. □

Theorem 2.3.6. *If f is a function defined on $[a,b]$ and $c \in (a,b)$, then f is integrable on $[a,b]$ if and only if it is integrable on $[a,c]$ and $[c,b]$. In this case*

$$\int_a^b f(t)\,dt = \int_a^c f(t)\,dt + \int_c^b f(t)\,dt.$$

Proof. This follows from the fact that the same things are true of the integrals of the real and imaginary parts g and h of f. □

Theorem 2.3.7. *If f is an integrable function on $[a,b]$, then*

$$\left|\int_a^b f(t)\,dt\right| \le \int_a^b |f(t)|\,dt.$$

Proof. This is proved using a trick. We set $w = \int_a^b f(t)\,dt$. If $w = 0$, there is nothing to prove. If $w \neq 0$, let $u = \overline{w}/|w|$. Then $uw = |w|$ and so

$$\left|\int_a^b f(t)\,dt\right| = u\int_a^b f(t)\,dt = \int_a^b uf(t)\,dt.$$

Since this is a real number, the integral of the imaginary part of uf is zero and we have

$$\left|\int_a^b f(t)\,dt\right| = \int_a^b \mathrm{Re}(uf(t))\,dt \leq \int_a^b |uf(t)|\,dt = \int_a^b |f(t)|\,dt. \qquad \square$$

A complex-valued function which is defined and continuous on an interval $[a,b]$ is clearly Riemann integrable on $[a,b]$, since its real and imaginary parts are continuous, and continuous real-valued functions on closed, bounded intervals are Riemann integrable.

Integration Along a Path. If γ is a path, then γ' exists and is continuous on each interval $[a_{i-1}, a_i]$ in a partition $a = a_0 < a_1 < \cdots < a_n = b$ of the parameter interval $[a,b]$. At the points $a_1, a_2, \cdots, a_{n-1}$ the definition of γ' is ambiguous – $\gamma'(a_j)$ has one value from the derivative of γ on $[a_{j-1}, a_j]$ and another from the derivative of γ on $[a_j, a_{j+1}]$. In order to remove this ambiguity, we choose to define γ' so as to be left continuous at these points. That is, at a_j, we choose the value for γ' that comes from its definition on $[a_{j-1}, a_j]$. Then γ' is well defined on $I = [a,b]$.

If f is a complex-valued function defined and continuous on a set E containing $\gamma(I)$, then the function $f(\gamma(t))\gamma'(t)$ is a well-defined function on I which is piecewise continuous in the following sense: It is continuous everywhere on $[a,b]$ except at the partition points $a_1, a_2, \cdots, a_{n-1}$. It is left continuous at these points, and the limit from the right exists and is finite at these points as well. In other words, this function is continuous from the left everywhere on $[a,b]$ and continuous except at finitely many points where it has simple jump discontinuities.

A function of this type is Riemann integrable on $[a,b]$. To see this, first observe that it is Riemann integrable on each subinterval $[a_{j-1}, a_j]$ because, on such an interval, the function agrees with a continuous function except at one point, a_{j-1}. A continuous function on a closed interval is Riemann integrable and changing its value at one point does not effect this fact or the value of the integral. Furthermore, by Theorem 2.3.6, if a function is Riemann integrable on two contiguous intervals, then it is integrable on their union. It follows that a function which is integrable on each subinterval in a partition of $[a,b]$ will be integrable on $[a,b]$.

The above discussion settles the question of the Riemann integrability of the integrand in the following definition.

Definition 2.3.8. Let $\gamma : [a,b] \to \mathbb{C}$ be a path and let f be a function which is defined and continuous on a set E which contains $\gamma([a,b])$. Then we define the integral of f over γ to be

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt. \tag{2.3.3}$$

One may think of this definition in the following way: the contour integral on the left in (2.3.3) is defined to be the Riemann integral obtained by replacing z by $\gamma(t)$ and dz by $\gamma'(t)dt$ and integrating over the parameter interval for γ.

In practice, we will calculate contour integrals by breaking the path up into its smooth sections, calculating the integrals over these sections and then adding the results. That this is legitimate follows from the fact that the Riemann integral of a function over the union of two contiguous intervals on the line is the sum of the integrals over the two intervals.

Examples.

Example 2.3.9. Find $\int_\gamma z\,dz$ if γ is the circular path defined in Example 2.3.2.

Solution: By Example 2.3.2, we have $\gamma(t) = r\,\mathrm{e}^{it}$ for $0 \le t \le 2\pi$ and $\gamma'(t) = ir\,\mathrm{e}^{it}$. Thus,

$$\begin{aligned}\int_\gamma z\,dz &= \int_0^{2\pi} r\,\mathrm{e}^{it}\, ir\,\mathrm{e}^{it}\;dt = ir^2 \int \mathrm{e}^{2it}\;dt = ir^2 \int_0^{2\pi} (\cos 2t + i\sin 2t)\,dt \\ &= ir^2 \int_0^{2\pi} \cos 2t\,dt - r^2 \int_0^{2\pi} \sin 2t\,dt = 0.\end{aligned}$$

Example 2.3.10. Find a path γ which traces the straight line from 0 to i followed by the straight line from i to $i+1$. Then calculate $\int_\gamma z^2\,dz$ for this path γ.

Solution: We may choose γ to be the path parameterized on $[0,2]$ as follows:

$$\gamma(t) = \begin{cases} it, & \text{if } 0 \le t \le 1; \\ i+t-1, & \text{if } 1 \le t \le 2. \end{cases}$$

We calculate the integrals over each of the two smooth sections of the path. On $[0,1]$ we have $(\gamma(t))^2 = -t^2$ and $\gamma'(t) = i$. Thus, the integral over the first section of the path is

$$\int_0^1 (\gamma(t))^2\gamma'(t)\,dt = \int_0^1 -t^2 i\,dt = -t^3 i/3\big|_0^1 = -i/3.$$

On $[1,2]$ we have $(\gamma(t))^2 = t^2 - 2t + 2(t-1)i$ and $\gamma'(t) = 1$. Thus, the integral over the second section of the path is

$$\int_1^2 (\gamma(t))^2\gamma'(t)\,dt = \int_0^1 (t^2-2t+2(t-1)i)\,dt = (t^3/3 - t^2 + (t^2-2t)i)\big|_1^2 = -2/3 + i.$$

Thus, $\int_\gamma z^2\,dz = -i/3 - 2/3 + i = -2/3 + 2i/3$.

Example 2.3.11. Find a path γ which traces once around the triangle with vertices $0, 1, i$ in the counterclockwise direction, starting at 0. For this path γ, find $\int_\gamma \overline{z}\,dz$.

Solution: A path γ with the required properties has parameter interval $[0,3]$ and is given by

$$\gamma(t) = \begin{cases} t, & \text{if } 0 \le t \le 1; \\ 2-t+(t-1)i & \text{if } 1 \le t \le 2; \\ (3-t)i & \text{if } 2 \le t \le 3. \end{cases}$$

On the interval $[0,1]$, we have $\overline{\gamma(t)} = t$ and $\gamma'(t) = 1$. Hence,

$$\int_0^1 \overline{\gamma(t)}\gamma'(t)\,dt = \int_0^1 t\,dt = 1/2.$$

On the interval $[1,2]$, we have $\overline{\gamma(t)} = 2 - t - (t-1)i$ and $\gamma'(t) = -1 + i$. Hence,

$$\int_1^2 \overline{\gamma(t)}\gamma'(t)\,dt = \int_1^2 (2t - 3 + i)\,dt = i.$$

On the interval $[2,3]$, we have $\overline{\gamma(t)} = t - 3$ and $\gamma'(t) = -i$. Hence,

$$\int_0^1 \overline{\gamma(t)}\gamma'(t)\,dt = \int_2^3 (3-t)i\,dt = i/2.$$

If we add the contributions of each of these three intervals, the result is

$$\int_\gamma \overline{z}\,dz = 1/2 + i + i/2 = 1/2 + (3/2)i.$$

Exercise Set 2.3

1. Find $\int_0^\pi e^{it}\,dt$.
2. Find $\int_0^1 \sin(it)\,dt$.
3. Find $\int_0^{2\pi} e^{int}\,e^{imt}\,dt$ for all integers n and m.
4. Find a path which traces the straight line joining $2 - i$ to $-1 + 3i$.
5. If $z_0 \in \mathbb{C}$, find a path which traces the circle of radius r, centered at z_0, (a) once in the counterclockwise direction, (b) once in the clockwise direction, (c) three times in the counterclockwise direction.
6. Prove that if $\gamma(t) = x(t) + iy(t)$ is a curve defined on an interval I, with real and imaginary parts $x(t)$ and $y(t)$, and if $c \in I$, then $\gamma'(c)$ exists if and only if $x'(c)$ and $y'(c)$ exist and, in this case, $\gamma'(c) = x'(c) + iy'(c)$.
7. Show that if f is a smooth complex-valued function on an interval $[a,b]$, then $\int_a^b f'(t)\,dt = f(b) - f(a)$.
8. Suppose γ is a path with parameter interval $[a,b]$. Use the result of the previous exercise to show that $\int_\gamma 1\,dz = \gamma(b) - \gamma(a)$.
9. Find $\int_\gamma z^2\,dz$ if γ traces a straight line from 0 to w.
10. Find $\int_\gamma z^{-1}\,dz$ and $\int_\gamma \overline{z}\,dz$ for the circular path $\gamma(t) = 3\,e^{it}$, $0 \le t \le 2\pi$.
11. Find $\int_\gamma \mathrm{Re}(z)\,dz$ if γ is the path of Example 2.3.11.
12. With γ as in the previous exercise, find $\int_\gamma \mathrm{Im}(z^2)\,dz$.
13. Is it generally true that $\mathrm{Re}(\int_\gamma f(z)\,dz) = \int_\gamma \mathrm{Re}(f(z))\,dz$?

2.4. Properties of Contour Integrals

We begin this section with the question of parameter independence. To what extent does the integral of a function along a path depend on how the path is parameterized? The same geometric figure $\gamma(I)$ may be parameterized in many ways. For example, the top third of the unit circle may be parameterized by

$$\begin{aligned} \gamma_1(t) &= -t + i\sqrt{1-t^2}, \quad -\sqrt{3}/2 \le t \le \sqrt{3}/2, \quad \text{or} \\ \gamma_2(t) &= \mathrm{e}^{it} = \cos t + i \sin t, \quad \pi/6 \le t \le 5\pi/6, \end{aligned} \tag{2.4.1}$$

and these are only two of infinitely many possibilities. Does the integral of a function over the upper third of the unit circle depend on which of these parmeterizations is chosen?

Parameter Changes that Change the Integral. The following example shows that some changes of parameterization do change the integral.

Example 2.4.1. Find $\int_{\gamma_1} 1/z\, dz$ if $\gamma_1(t) = r\,\mathrm{e}^{it}$ on $[0, 2\pi]$ is the circular path of Example 2.3.2. Does the answer change if the circle is traversed in the clockwise direction instead, using the path $\gamma_2(t) = r\,\mathrm{e}^{-it}$ on $[0, 2\pi]$?

Solution: From Example 2.3.2 we know that the path $\gamma_1(t) = r\,\mathrm{e}^{it}$ has $\gamma_1'(t) = ir\,\mathrm{e}^{it}$ and so the given integral is

$$\int_{\gamma_1} \frac{dz}{z} = \int_0^{2\pi} \frac{(r\,\mathrm{e}^{it})'}{r\,\mathrm{e}^{it}} dt = \int_0^{2\pi} i\, dt = 2\pi i.$$

On the other hand, the derivative of $\gamma_2 = \mathrm{e}^{-it}$ is $-ir\,\mathrm{e}^{-it}$ and so

$$\int_{\gamma_2} \frac{dz}{z} = \int_0^{2\pi} -i\, dt = -2\pi i.$$

This example shows that the integral along a path depends not only on the geometric figure that is the image $\gamma(I)$ of the path, but also on the direction the path is traversed (at the very least).

Also, traversing a portion of the curve more than once may affect the integral. For example, if we were to go around the circle twice in Example 2.4.1, by choosing $\gamma(t) = \mathrm{e}^{2it}$ on $[0, 2\pi]$, the result would be $4\pi i$ instead of $2\pi i$.

The Independence of Parameterization Theorem. There is a degree to which the integral is independent of the parameterization. Certain ways of changing the parameterization do not effect the integral, as the following theorem shows.

Theorem 2.4.2. *Let $\gamma_1 : [a, b] \to \mathbb{C}$ be a path and $\alpha : [c, d] \to [a, b]$ a smooth function with $\alpha(c) = a$ and $\alpha(d) = b$. If γ_2 is the path with parameter interval $[c, d]$ defined by $\gamma_2(t) = \gamma_1(\alpha(t))$, then*

$$\int_{\gamma_2} f(z) dz = \int_{\gamma_1} f(z) dz$$

for every function f defined and continuous on a set E containing $\gamma_1([a, b]) = \gamma_2([c, d])$.

Proof. We have $\gamma_2(t) = \gamma_1(\alpha(t))$ and, by the chain rule,

$$\gamma_2'(t) = \gamma_1'(\alpha(t))\alpha'(t).$$

Thus,

$$\begin{aligned}\int_{\gamma_2} f(z)\,dz &= \int_c^d f(\gamma_2(t))\gamma_2'(t)\,dt \\ &= \int_c^d f(\gamma_1(\alpha(t)))\gamma_1'(\alpha(t))\alpha'(t)\,dt \\ &= \int_a^b f(\gamma_1(s))\gamma_1'(s)\,ds \\ &= \int_{\gamma_1} f(z)\,dz,\end{aligned}$$

where the third equality follows from the substitution $s = \alpha(t)$. This completes the proof. $\square$

Note that the condition that $\alpha(c) = a$ and $\alpha(d) = b$ is essential in the above theorem. It says that α takes the endpoints of the parameter inverval $[c, d]$ to the endpoints of the parameter interval $[a, b]$ in an order preserving fashion.

Example 2.4.3. Are the integrals of a continuous function over the two paths in (2.4.1) necessarily the same?

Solution: Yes. If we set $\alpha(t) = -\cos t$, then α is a smooth function mapping the parameter interval $[\pi/6, 5\pi/6]$ to the parameter interval $[-\sqrt{3}/2, \sqrt{3}/2]$ in an order preserving fashion. Furthermore, $\gamma_2 = \gamma_1 \circ \alpha$. Thus, the above theorem insures that the integral of a continuous function over γ_1 is the same as its integral over γ_2.

Doesn't Example 2.4.1 contradict Theorem 2.4.2? After all, if $\gamma_1(t) = \mathrm{e}^{it}$ on $[0, 2\pi]$ and $\alpha : [0, 2\pi] \to [0, 2\pi]$ is defined by $\alpha(t) = 2\pi - t$, then $\gamma_2(t) = \gamma_1(\alpha(t)) = \mathrm{e}^{-it}$. By Example 2.4.1 the integrals of $1/z$ over these two curves are different. Doesn't Theorem 2.4.2 say they should be the same? No. The conditions $\alpha(a) = c$ and $\alpha(b) = d$ are not satisfied by this choice of α, since $\alpha(0) = 2\pi$ and $\alpha(2\pi) = 0$. In other words, this choice of α reverses the order of the endpoints of the parameter interval rather than preserving that order.

In general, the conditions $\alpha(a) = c$ and $\alpha(b) = d$ guarantee that, overall, γ_2 traverses the curve in the same direction as γ_1. If α' were positive on the entire interval, then α would be increasing on this interval and γ_1 and γ_2 would be moving in the same direction at each point of the curve. If α' is not positive on all of $[c, d]$, then there may be intervals where one path reverses direction and backtracks, while the other path does not. These things do not affect the integral, because if a curve does backtrack for a time, it has to turn around and recover the same ground in order to catch up to the other curve in the end. This is an intuitive explanation; the actual proof that the integral is unaffected is in the proof of the above theorem.

Theorem 2.4.2 leads to a strategy which, for some paths γ_1 and γ_2 with the same image, yields a proof that they determine the same integral: Suppose that the parameter intervals for the two paths can each be partitioned into n subintervals in such a way that for $j = 1, \cdots, n$, γ_1 on its jth subinterval and γ_2 on its jth subinterval are related by a smooth function α_j, as in Theorem 2.4.2. If this can be done, then it clearly follows that $\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$ for any function f which is continuous on a set containing $\gamma_1(I)$. For this reason, Theorem 2.4.2 is sometimes called the *independence of parameterization theorem.*

Remark 2.4.4. Since path integrals are essentially independent of the way the path is parameterized, we will often describe a path without specifying a parameterization. Instead, we will just give a description of the geometric object that is traced, the direction, and how many times. For example, we may describe a path as tracing once around the unit circle in the counterclockwise direction, or tracing once around the boundary $\partial\Delta$ of a given triangle Δ in the counterclockwise direction, or as tracing the straight line path from a complex number w_1 to a complex number w_2. In the first two cases we may simply write

$$\int_{|z|=1} f(z)\,dz \quad \text{or} \quad \int_{\partial\Delta} f(z)\,dz$$

for the corresponding path integral. In the latter case, we may write

$$\int_{w_1}^{w_2} f(z)\,dz$$

for the path integral along the straight line from w_1 to w_2.

Closed Curves. The curves in Examples 2.3.2 and 2.3.4 both have the property that they begin and end at the same point – that is, they are closed curves. A closed curve γ on a parameter interval $[a, b]$ is one that satisfies $\gamma(a) = \gamma(b)$. A closed curve which is a path will be called a *closed path*.

The famous integral theorem of Cauchy states that the integral of an analytic function f around a closed path is 0, provided there is an appropriate relationship between the curve γ and the domain U on which f is analytic (roughly speaking, the curve should lie in U but not go around any *holes* in U). Since the function $f(z) = z$ is analytic on $\mathbb{C}$ (as is any polynomial in z), the next example illustrates this phenomenon.

Example 2.4.5. Find $\int_\gamma z\,dz$ if γ is the path of Example 2.3.4.

Solution: From Example 2.3.4 we know that the path $\gamma(t)$ has values $4t, 1 + (4t-1)i, 3-4t+i, (4-4t)i$ and derivatives $4, 4i, -4$, and $-4i$ on the four subintervals of the partition $0 < 1/4 < 1/2 < 3/4 < 1$. Thus, the integrals over the four smooth pieces of our curve are

$$\int_0^{1/4} 4t \cdot 4\,dt = 8t^2\Big|_0^{1/4} = 1/2,$$

$$\int_{1/4}^{1/2} (1 + (4t-1)i) \cdot 4i\,dt = (4ti - 8t^2 + 4t)\Big|_{1/4}^{1/2} = i - 1/2,$$

$$\int_{1/2}^{3/4} (3 - 4t + i) \cdot (-4)\,dt = (-12t + 8t^2 - 4ti)\Big|_{1/2}^{3/4} = -1/2 - i,$$

$$\int_{3/4}^{1} (4-4t)i \cdot (-4i)\,dt = (+16t - 8t^2)\Big|_{3/4}^{1} = 1/2.$$

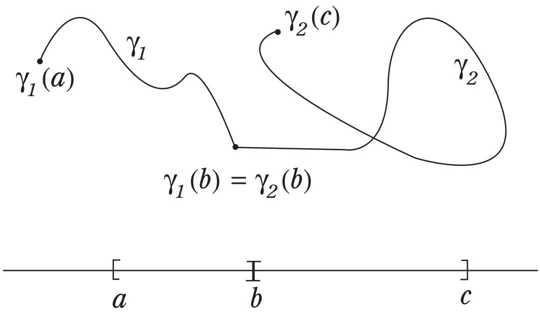

Figure 2.4.1. The Join of Two Paths.

Since these add up to 0, we have $\int_\gamma z\,dz = 0$.

The function $1/z$ is also analytic, except at $z = 0$. The circular path of Example 2.4.1 is closed and lies in the domain where $1/z$ is analytic. So why is the integral not 0? Because the path goes around a hole in the domain of $1/z$ – it goes around $\{0\}$.

Additivity Properties of Contour Integrals. If γ is a path with parameter interval $[a, b]$, then we can use Theorem 2.4.2 to change the parameter interval to any other interval $[c, d]$ with $c < d$, in a way that does not affect the image of γ or integrals over γ. In fact, if we set

$$\alpha(t) = a + \frac{b-a}{d-c}(t-c),$$

then α is smooth, $\alpha([c, d]) = [a, b]$, $\alpha(c) = a$ and $\alpha(d) = b$. Thus, $\gamma_1(t) = \gamma(\alpha(t))$ defines a path γ_1 with the same image as γ and, by Theorem 2.4.2, a path which determines the same integral for continuous functions on its image. Thus, without loss of generality, we may always assume that the parameter interval for a path is any interval we choose.

If γ_1 and γ_2 are two paths so that γ_1 ends where γ_2 begins, then we can join the two paths to form a single new path $\gamma_1 + \gamma_2$. We do this as follows: If γ_1 has parameter interval $[a, b]$, we choose a parameter interval of the form $[b, c]$ for γ_2. The fact that γ_2 begins where γ_1 ends means that $\gamma_1(b) = \gamma_2(b)$. We define $\gamma_1 + \gamma_2$ on $[a, c]$ by

$$(\gamma_1 + \gamma_2)(t) = \begin{cases} \gamma_1(t) & \text{if } \ t \in [a, b], \\ \gamma_2(t) & \text{if } \ t \in [b, c]. \end{cases} \tag{2.4.2}$$

The path $\gamma_1 + \gamma_2$ is called the *join* of γ_1 and γ_2.

In Example 2.4.1, changing the path from one tracing the circle counterclockwise to one tracing the circle clockwise had the effect of changing the sign of the integral. As we shall see, this always happens. If $\gamma : [a, b] :\to \mathbb{C}$ is a path, denote by $-\gamma$ the path defined by

$$-\gamma(t) = \gamma(a + b - t).$$

Then $-\gamma(a) = \gamma(b)$ and $-\gamma(b) = \gamma(a)$. In fact, $-\gamma$ traces the same geometric figure as γ, but it does so in the opposite direction.

For some closed curves, such as circles, and boundaries of rectangles, triangles, etc., there is clearly a clockwise direction around the curve and a counterclockwise direction. If such a curve is parameterized so that it is traversed in the counterclockwise direction, we will say the resulting closed path has *positive orientation.* If it is traversed in the clockwise direction, we will say it has *negative orientation.* Clearly, if γ has positive orientation, then $-\gamma$ has negative orientation.

The common starting and ending point of a closed path can be changed without changing the integral of a function over this path. This is done by representing the closed path as the join of two paths which connect the original starting and ending point to the new one. One then uses part (b) of the next theorem. The details are left to the exercises.

The next theorem states the elementary properties of path integrals having to do with linearity and path additivity. Part (b) follows immediately from the corresponding additivity property of the Riemann integral on the line and we have already used it several times. We leave the proofs of (a) and (c) to the exercises (Exercise 2.4.5).

Theorem 2.4.6. *Let $\gamma, \gamma_1, \gamma_2$ be paths with γ_1 ending where γ_2 begins, f and g two functions which are continuous on a set E containing the images of these paths, and a and b complex numbers. Then*

(a) $\displaystyle\int_\gamma (af(z) + bg(z))\,dz = a\int_\gamma f(z)\,dz + b\int_\gamma g(z)\,dz;$

(b) $\displaystyle\int_{\gamma_1+\gamma_2} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz;$

(c) $\displaystyle\int_{-\gamma} f(z)\,dz = -\int_\gamma f(z)\,dz.$

Part (a) of this theorem says that a path integral is a linear function of the integrand, Part (b) says that it is an additive function of the path, while Part (c) shows why the notation $-\gamma$ is appropriate for the curve that is γ traversed in the opposite direction.

Length of a Path. We define the length $\ell(\gamma)$ of a path γ in $\mathbb{C}$ in the same way the length of a curve in $\mathbb{R}^2$ is defined in calculus.

Definition 2.4.7. If $\gamma(t) = x(t) + iy(t)$ is a path in $\mathbb{C}$ with parameter interval $[a, b]$, then the length $\ell(\gamma)$ of γ is defined to be

$$\ell(\gamma) = \int_a^b |\gamma'(t)|\,dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt.$$

Example 2.4.8. Prove that the above definition of length yields the correct length for a path which traces once around a circle of radius r.

Solution: The path is $\gamma(t) = r\,e^{it}$, with parameter interval $[0, 2\pi]$. The derivative of γ is $\gamma'(t) = ir\,e^{it}$ and so $|\gamma'(t)| = r$. Thus, $\ell(\gamma) = \int_0^{2\pi} r\,dt = 2\pi r$.

It will be important in coming sections to be able to obtain good upper bounds on the absolute value of a path integral. The key theorem that produces such upper bounds is the following.

Theorem 2.4.9. *Let γ be a path in $\mathbb{C}$ and f a function continuous on a set containing $\gamma(I)$. If $|f(z)| \le M$ for all $z \in \gamma(I)$, then*

$$\left| \int_\gamma f(z)\, dz \right| \le M\ell(\gamma).$$

Proof. If the parameter interval for γ is $[a, b]$, then

$$\begin{aligned} \left| \int_\gamma f(z)\, dz \right| &= \left| \int_a^b f(\gamma(t))\gamma'(t)\, dt \right| \le \int_a^b |f(\gamma(t))\gamma'(t)|\, dt \\ &\le \int_a^b M|\gamma'(t)|\, dt = M \int_a^b |\gamma'(t)|\, dt = M\ell(\gamma). \end{aligned}$$ □

The next example is a typical application of this theorem.

Example 2.4.10. Show that if f is a bounded continuous function on $\mathbb{C}$, and γ_R is the path $\gamma_R(z) = R\,e^{it}$ for $t \in [0, 2\pi]$, then

$$\lim_{R\to\infty} \int_{\gamma_R} \frac{f(z)}{(z-w)^2}\, dz = 0 \tag{2.4.3}$$

for each $w \in \mathbb{C}$.

Solution: The statement that f is bounded means there is an upper bound M for $|f|$. That is, $|f(z)| \le M$ for all $z \in \mathbb{C}$. We also have $|z - w| \ge |z| - |w|$ by the second form of the triangle inequality. If $z \in \gamma(I)$, then $|z| = R$ and so $|z - w| \ge R - |w|$, which implies $|z - w|^{-2} \le (R - |w|)^{-2}$. Thus, for $z \in \gamma(I)$, we have the following bound on the integrand of (2.4.3):

$$\left| \frac{f(z)}{z-w} \right| \le \frac{M}{(R-|w|)^2}.$$

Since $\ell(\gamma_R) = 2\pi R$, Theorem 2.4.9 implies that

$$\left| \int_{\gamma_R} \frac{f(z)}{(z-w)^2}\, dz \right| \le \frac{2\pi MR}{(R-|w|)^2}.$$

The right side of this inequality has limit 0 as $R \to \infty$ and this implies (2.4.3).

Exercise Set 2.4

1. Compute $\int_\gamma z^2\, dz$ if γ is any path which traces once around the circle of radius one in the counterclockwise direction.
2. Compute $\int_\gamma 1/z\, dz$ if γ is any path which traces twice around the circle of radius one, centered at 0, in the counterclockwise direction.
3. If z_0 and w_0 are two points of $\mathbb{C}$, compute $\int_\gamma z\, dz$ if γ is any path which traces the straight line from z_0 to w_0 once.

4. Compute the integral of the previous exercise for any smooth path γ which begins at z_0 and ends at w_0.
5. Prove Parts (a) and (c) of Theorem 2.4.6.
6. Describe a smooth, order preserving function α which takes the parameter interval $[0, 1]$ to the parameter interval $[2, 5]$.
7. Prove that a parameter change $\gamma \to \gamma \circ \alpha$, like the one in Theorem 2.4.2, does not change the length of a path provided α is an non-decreasing function (has a non-negative derivative).
8. Show that $\left| \int_\gamma \frac{\cos z}{z} \, dz \right| \leq 2\pi e$ if γ is a path that traces the unit circle once. Hint: Show that $|\cos z| \leq e$ if $|z| = 1$.
9. Show that if Δ is a triangle in the plane of diameter d (length of its longest side), and if f is a continuous function on Δ with $|f|$ bounded by M on Δ, then
$$\left| \int_{\partial \Delta} f(z) \, dz \right| \leq 3Md.$$
10. Prove that $\int_\gamma p(z) \, dz = 0$ if $\gamma(t) = \mathrm{e}^{it}$, $0 \leq t \leq 2\pi$, and $p(z)$ is any polynomial in z (this is a special case of Cauchy's Theorem, but do not assume Cauchy's Theorem in your proof).
11. Let $R(z)$ be the remainder after n terms in the power series for e^z. That is,
$$R(z) = \mathrm{e}^z - \sum_{k=1}^{n} \frac{z^k}{k!} = \sum_{k=n+1}^{\infty} \frac{z^k}{k!}.$$
Prove that $|R(z)| \leq \dfrac{e-1}{(n+1)!}$ if $|z| \leq 1$.
12. Prove that $\int_\gamma \mathrm{e}^z \, dz = 0$ if $\gamma(t) = \mathrm{e}^{it}$, $0 \leq t \leq 2\pi$ using the previous exercise and Exercise 10.
13. Prove that if γ is a closed path with parameter interval $I = [a, b]$ and common starting and ending point $z = \gamma(a) = \gamma(b)$ and w is any other point on $\gamma(I)$, then there is another closed path γ_1 with $\gamma_1(I) = \gamma(I)$, which determines the same integral, but has w as common starting and ending point. Hint: Use part (b) of Theorem 2.4.6.

2.5. Cauchy's Integral Theorem for a Triangle

The core material of any beginning Complex Variables text is the proof of Cauchy's Integral Theorem and the exploration of its consequences. Roughly speaking, Cauchy's Integral Theorem states that the integral of an analytic function around a closed path is zero, provided the path is contained in the open set U on which the function is analytic and does not go around any "holes" in U. Part of the problem here is to make sense of the idea of a "hole" in an open set and to decide what it means for a path to go around such a hole.

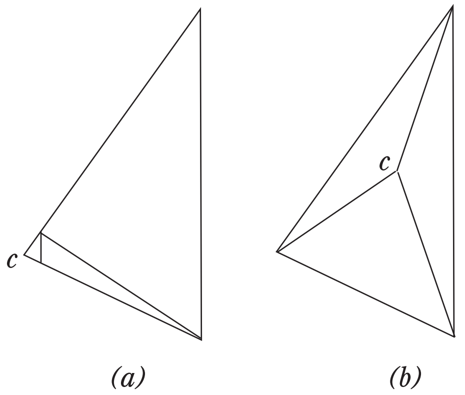

Figure 2.5.3. Dealing with an Exceptional Point c.

of each of these is zero, so the integral around $\partial\Delta$ is also 0. This completes the proof. □

Exercise Set 2.5

1. Prove the Bolzano-Weierstrass Theorem: If K is a compact subset of $\mathbb{R}^n$, then every sequence in K has a subsequence which converges to an element of K.
2. Use Corollary 2.5.5 to show that if K is a compact subset of $\mathbb{C}$ and f is a continuous complex-valued function on K, then the modulus $|f(z)|$ of f takes on a maximal value at some point of K.
3. Show that if K is a compact subset of $\mathbb{C}$, then there is a point $z_0 \in K$ of minimum modulus – that is, a point $z_0 \in K$ such that
$$|z_0| \leq |z| \quad \text{for all} \quad z \in K.$$
4. Prove that if g is analytic on an open subset U of $\mathbb{C}$ and $\gamma : [a, b] \to U$ is a path in U, then
$$(g(\gamma(t)))' = g'(\gamma(t))\gamma'(t)$$
for $t \in [a, b]$. Hint: The proof is very similar to the proof of Theorem 2.2.7.
5. Calculate $\int_\gamma z^n\, dz$ if n is a non-negative integer and γ is a path in the plane joining the point z_0 to the point w_0. Hint: Use Theorem 2.5.6.
6. Show that $\int_\gamma p(z)\, dz = 0$ if γ is any closed path in the plane and p is any polynomial.
7. Calculate $\int_\gamma 1/z\, dz$ if γ is any path in $\mathbb{C}$ joining $-i$ to i which does not cross the half-line $(-\infty, 0]$ on the real axis. Hint: Use the result of Example 2.2.11 and Theorem 2.5.6.

8. Using the same hint as in the previous exercise, show that
$$\int_\gamma \frac{1}{z}\,dz = 0$$
if γ is any closed path contained in the complement of the set of non-positive real numbers. Compare this with Example 2.4.1.

9. If $\sqrt{z}$ is defined by $\sqrt{z} = e^{(\log z)/2}$ for the branch of the log function defined by the condition $-\pi/2 \le \arg(z) \le 3\pi/2$, find an antiderivative for $\sqrt{z}$ and then find $\int_\gamma \sqrt{z}\,dz$, where γ is any path from -1 to 1 which lies in the upper half-plane.

10. Prove that if f is analytic in an open set containing a rectangle R, then the path integral of f around the boundary of this rectangle is 0.

11. Let γ be the path which traces the straight line from 1 to $1+i$, then the straight line from $1+i$ to i and then the straight line from i to 0. Calculate $\int_\gamma z^n\,dz$.

12. Let Δ be the triangle with vertices $1-i, i,$ and $-1-i$ and S be the square with vertices $1-i, 1+i, -1+i,$ and $-1-i$. If f is any function which is analytic on $\mathbb{C}\setminus\{0\}$, prove that
$$\int_{\partial\Delta} f(z)\,dz = \int_{\partial S} f(z)\,dz,$$
where $\partial\Delta$ and ∂S are traversed in the counterclockwise direction.

13. For any pair of points a, b in $\mathbb{C}$, denote the integral of a function f along the straight line segment joining a to b by $\int_a^b f(z)\,dz$, as in Remark 2.4.4. Suppose f is analytic in an open set containing the triangle with vertices a, b, c. Show that
$$\int_a^c f(z)\,dz - \int_a^b f(z)\,dz = \int_b^c f(z)\,dz.$$

14. Show that Theorem 2.5.9 can be strengthened to conclude that the integral of f around any triangle in U is 0 if f is continuous on U and analytic on $U\setminus I$, where I is an interval contained in U. Hint: First consider the case where one side of the triangle lies along the interval I.

15. If f is analytic on an open set U, then the integral of f around the boundary of any triangle in U is 0 (Theorem 2.5.8), as is its integral around the boundary of any rectangle in U (Exercise 2.5.10). What other geometric figures have this property? What is the most general theorem along these lines you can think of?

2.6. Cauchy's Theorem for a Convex Set

A convex set C in $\mathbb{C}$ is a set with the property that if a and b are points in C, then the line segment joining a and b is also contained in C.

Existence of Antiderivatives. The strategy for proving Cauchy's Theorem for convex sets is to prove that every analytic function on a convex set has an antiderivative and then apply Theorem 2.5.6. The first step in this program is accomplished with the following theorem.

even if the function is not analytic at some point but is continuous there, it follows that

$$0 = \int_\gamma g(z,w)dw = \int_\gamma \frac{f(w)}{w-z}dw - \int_\gamma \frac{f(z)}{w-z}dw$$
$$= \int_\gamma \frac{f(w)}{w-z}dw - 2\pi i \operatorname{Ind}_\gamma(z) f(z),$$

as long as z is not on the contour γ (note that this is required in order to write the integral in the first line above as the difference of two integrals, since otherwise these two integrals might not exist individually). We conclude that

$$\operatorname{Ind}_\gamma(z) f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}dw,$$

as required. This completes the proof. □

This is a striking result, for it says that the values of an analytic function at points "inside" a closed path are determined by its values at points on the path. Here, a point is considered inside the path if the path has non-zero index at the point.

Corollary 2.6.8. *If U is a convex open set, $z \in U$ and γ is a closed path in U with* $\operatorname{Ind}_\gamma(z) = 1$, *then*

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw$$

for every function f analytic on U.

Intuitively, the meaning of the hypothesis $\operatorname{Ind}_\gamma(z) = 1$ in the above corollary is that the closed path γ goes around z once and does so in the positive direction.

Cauchy's Integral Theorem and Cauchy's Integral Formula have a wealth of applications. We will begin exploring these in the next chapter.

However, in order for Cauchy's Integral Theorem, in the above form, to be usable, we need to be able to easily compute the index of a curve around a given point. The last section of this chapter is devoted to developing the essential properties of the index function which make this possible.

Exercise Set 2.6

1. Prove that a function which has complex derivative identically 0 on a convex open set U is constant on U.
2. Calculate $\int_\gamma (z^2-4)^{-1}\,dz$ if γ is the unit circle traversed once in the positive direction.
3. Calculate $\int_\gamma (1-e^z)^{-1}\,dz$ if γ is the circle $\gamma(t) = 2i + e^{it}$.
4. Calculate $\int_\gamma 1/z\,dz$ if γ is any circle which does not pass through 0. Note that the answer depends on γ.
5. Find $\int_\gamma 1/z^2\,dz$ if γ is any closed path in $\mathbb{C}\setminus\{0\}$.

6. Show that the principal branch of the log function can be described by the formula $\log(z) = \int_1^z 1/w\,dw$ for $z \notin (-\infty, 0]$.
7. Prove Corollary 2.6.3.
8. Without doing any calculating, show that the integral of $1/z$ around the boundary of the triangle with vertices $i, 1-i, -1-i$ is $2\pi i$.
9. Let f be a function which is analytic on $\mathbb{C}\backslash\{z_0\}$. Show that the contour integral of f around a circle of radius $r > 0$, centered at z_0, is independent of r.
10. Calculate $\mathrm{Ind}_\gamma(z_0)$ if $\gamma(t) = z_0 + \mathrm{e}^{int}, t \in [0, 2\pi]$ and n is any integer.
11. Calculate $\mathrm{Ind}_\gamma(1+i)$ if γ is the path which traces the line from 0 to 2, then proceeds counterclockwis around the circle $|z| = 2$ from 2 to $2i$ and then traces the line from $2i$ to 0. What is the answer if this path is traversed in the opposite direction?
12. Use Cauchy's Integral Formula to calculate $\displaystyle\int_{|z|=1} \frac{\mathrm{e}^z}{z}\,dz$.
13. Use Cauchy's Formula to show that
$$\int_{|z-1|=1} \frac{1}{z^2-1}\,dz = \pi i, \quad \int_{|z+1|=1} \frac{1}{z^2-1}\,dz = -\pi i.$$
14. Show that
$$\int_{|z|=3} \frac{1}{z^2-1}\,dz = 0.$$
Hint: Use the result of the preceding exercise.
15. Use Cauchy's Integral Formula to prove that if f is a function which is analytic in an open set containing the closed unit disc $\overline{D}_1(0)$, and if $T = \{z : |z| = 1\}$ is the unit circle, then $|f(0)| \leq M$, where M is the maximum value of $|f|$ on T.
16. Show that if γ is a path from z_1 to z_2 which does not pass through the point z_0, then
$$\int_\gamma \frac{1}{w-z_0}\,dw = \log\left(\frac{z_2-z_0}{z_1-z_0}\right),$$
for some branch of the log function. Note that, in the case where $z_1 = z_2$, this is just Theorem 2.6.6.

2.7. Properties of the Index Function

If γ is a closed path, then removing $\gamma(I)$ from the plane results in a set which is divided into a number of connected pieces. These are open sets called the *connected components* of the complement of $\gamma(I)$. We will prove that $\mathrm{Ind}_\gamma(z)$ is constant on each of these components. Thus, to calculate $\mathrm{Ind}_\gamma(z)$ on a given component, one only needs to calculate it at one point of the component.

Before proving this, we need to have a firm idea of what a connected component is. This leads to a discussion of connected sets.