PROBLEM 1

(a) Find the Fourier integral representation of §7.1 #1

$$f(x) = \begin{cases} 1 & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

(b) Use (a) to find

$$\int_0^\infty \frac{\sin(\omega)\cos(\omega/2)}{\omega} d\omega. \quad (5\text{ pt}) \quad \#13(a)$$

(a) $A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t) \cos\omega t \, dt = \frac{1}{\pi}\int_{-1}^{1} \cos\omega t \, dt =$

$= \left[\frac{\sin \omega t}{\pi \omega} \right]_{-1}^{1} = \frac{2\sin\omega}{\pi\omega}$ (5 pt) (if $\omega \neq 0$)

($\boxed{\omega = 0}$: $A(0) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\,dt = \frac{2}{\pi} = \lim_{\omega \to 0} A(\omega)$)

(5 pt)

$B(\omega) = 0$ as $f(x)$ is even (5 pt) No such x !!

$$\Rightarrow \frac{f(x+)+f(x-)}{2} = \int_0^\infty \left(\frac{2\sin\omega}{\pi\omega}\right) \cos\omega x \, d\omega$$

(b) plug in $x = \frac{1}{2} \Rightarrow$

$$1 = f\left(\tfrac{1}{2}\right) = \frac{f(\frac{1}{2}^+) + f(\frac{1}{2}^-)}{2} = \int_0^\infty \frac{2\sin\omega}{\pi\omega} \cos\frac{\omega}{2} \, d\omega$$

$$\Rightarrow \boxed{\frac{\pi}{2}} = \int_0^\infty \frac{\sin\omega}{\omega} \cos\frac{\omega}{2} \, d\omega$$

PROBLEM 2

Here and after, assume that all functions are absolutely integrable and have Fourier transforms.

(a) Prove that $\mathcal{F}(f') = (i\omega)\mathcal{F}(f)$.
(b) If $\mathcal{F}(e^{ix}f(x))(\omega) = \omega - i2$ and $\mathcal{F}(e^{-ix}f(x))(\omega) = \omega^2 - i2$, find $\mathcal{F}(\cos(x)f(x))(\omega)$. (§7.2 #20)

(a) 2 ways (either is fine) (5pt)

$\boxed{1st}$: $\mathcal{F}(f')(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'(x)e^{-i\omega x}dx$

integ. by parts $\frac{1}{\sqrt{2\pi}}\left\{\underbrace{\left[f(x)e^{-i\omega x}\right]_{-\infty}^{\infty}}_{(5pt)} - \underbrace{(-i\omega)\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx}_{(5pt)}\right\}$

$= 0 + i\omega\,\mathcal{F}(f)$. (5pt)

$\boxed{2nd}$ $f(x) = \mathcal{F}^{-1}(\hat{f}(\omega)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega x}d\omega$

$f'(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\hat{f}(\omega)\,i\omega\right)e^{i\omega x}d\omega = \mathcal{F}^{-1}(\hat{f}(\omega)\,i\omega)$

$\mathcal{F}(f'(x)) = \mathcal{F}\circ\mathcal{F}^{-1}(\hat{f}(\omega)\,i\omega) = \hat{f}(\omega)(i\omega)$

(b) $\mathcal{F}(\cos(x)f(x)) = \mathcal{F}\left(\frac{e^{ix}+e^{-ix}}{2}f(x)\right) =$

$= \frac{1}{2}\mathcal{F}(e^{ix}f(x)) + \frac{1}{2}\mathcal{F}(e^{-ix}f(x))$ (5pt)

$= \frac{1}{2}(\omega - i2) + \frac{1}{2}(\omega^2 - i2) = \frac{\omega+\omega^2}{2} - i2$

(5pt)

PROBLEM 3

Consider the following heat equation problem

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty, \qquad t > 0,$$
$$u(x,0) = f(x),$$

(a) Find $\hat{u}(\omega, t)$ in terms of $\hat{f}(\omega)$. Show all steps.
(b) Find $u(x,t)$ in (a) for $f(x) = e^{-x^2/2}$. You may use the following facts

$$\mathcal{F}\left(e^{-\frac{x^2}{2}}\right) = e^{-\frac{\omega^2}{2}}, \qquad \mathcal{F}^{-1}\left(e^{-\frac{\omega^2}{2a}}\right) = \sqrt{a}e^{-\frac{ax^2}{2}}.$$

(§7.3 #3)

(a) Apply Fourier transf to both eq'ns:

$$\begin{cases} \frac{d}{dt}\hat{u}(\omega,t) = -c^2\omega^2\hat{u}(\omega,t) & \text{--- (1)} \\ \hat{u}(\omega,0) = \hat{f}(\omega) & \text{--- (2)} \end{cases}$$

$(1) \Rightarrow \hat{u}(\omega,t) = A(\omega)\, e^{-c^2\omega^2 t}$

$(2) \Rightarrow \hat{u}(\omega,0) = A(\omega)\cdot 1 = \hat{f}(\omega)$

$\hat{u}(\omega,t) = \hat{f}(\omega)\, e^{-c^2\omega^2 t}$

(b) $\hat{u}(\omega,t) = \hat{f}(\omega)\, e^{-c^2\omega^2 t}$

$$\left(\begin{aligned} &f(x) = e^{-\frac{x^2}{2}} \\ &\mathcal{F}(f(x)) = \hat{f}(\omega) = e^{-\frac{\omega^2}{2}} \end{aligned} \right)$$

$= e^{-\frac{\omega^2}{2}}\, e^{-c^2\omega^2 t}$

$= e^{-(c^2t+\frac{1}{2})\,\omega^2}$

$u(x,t) = \mathcal{F}^{-1}(\hat{u}(\omega,t)) = \mathcal{F}^{-1}\left(e^{-(c^2t+\frac{1}{2})\,\omega^2}\right)$

$\frac{1}{2a} = c^2t + \frac{1}{2} \Rightarrow a = (2c^2t+1)^{-1}$

$$= \frac{1}{\sqrt{2c^2t+1}}\, e^{-\frac{x^2}{4c^2t+2}}$$