MT3 Soln

PROBLEM 1

Consider the following heat boundary value problem

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L, \qquad t > 0,$$
$$u(0,t) = T_1, \quad u(L,t) = T_2, \qquad t > 0,$$
$$u(x,0) = f(x), \qquad 0 < x < L.$$

Set $T_1 = 0, T_2 = 0, L = \pi, c = 1, f(x) =$ [redacted]. $30 \sin x$

Solve this problem without using any formulas from Chapter 3 of the textbook.

- SOV $\quad u(x,t) = X(x)\, T(t)$

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \Rightarrow X(x)\, T'(t) = c^2 X''(x) T(t) \Rightarrow \frac{T'(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = k$$

- For $X(x)$: $\quad \begin{cases} X'' - kX = 0 \\ X(0) = 0 \\ X(\pi) = 0 \end{cases} \Rightarrow$ char. eqn $\lambda^2 - k = 0$

① $k > 0 \Rightarrow \lambda = \pm\sqrt{k}$, $\quad X(x) = c_1 e^{\sqrt{k}x} + c_2 e^{-\sqrt{k}x}$

b.c. $\begin{cases} X(0) = 0 \Rightarrow c_1 + c_2 = 0 \Rightarrow c_1 = -c_2 \\ X(\pi) = 0 \Rightarrow \underbrace{c_1}_{=-c_2} e^{\sqrt{k}\pi} + c_2 e^{-\sqrt{k}\pi} = 0 \Rightarrow \left(-c_2 \underbrace{e^{\sqrt{k}\pi}}_{\neq 0}\right)\left(\underbrace{1 - e^{-2\sqrt{k}\pi}}_{\neq 0}\right) = 0 \end{cases}$

$\Rightarrow c_2 = 0 \Rightarrow c_1 = 0 \Rightarrow X = 0$

② $k = 0 \Rightarrow X'' = 0 \Rightarrow X(x) = c_1 x + c_2$,

b.c. $\begin{cases} X(0) = 0 \Rightarrow 0 + c_2 = 0 \Rightarrow c_2 = 0 \\ X(\pi) = 0 \Rightarrow c_1 \pi = 0 \Rightarrow c_1 = 0 \end{cases} \Rightarrow X = 0$

③ $k < 0$ $\quad$ Let $k = -\mu^2 \Rightarrow X(x) = c_1 \cos \mu x + c_2 \sin \mu x$.

b.c $\begin{cases} X(0) = 0 \Rightarrow c_1 + 0 = 0 \Rightarrow c_1 = 0 \\ X(\pi) = 0 \Rightarrow c_2 \sin \mu\pi = 0 \Rightarrow \mu\pi = n\pi \Rightarrow \mu = n. \end{cases}$

$\Rightarrow \boxed{X_n(x) = \sin(nx)}$ $\qquad$ for any n

Prob 1 (continued)

- **For $T(t)$**: $T' - kc^2T = T' + n^2 T = 0 \Rightarrow T_n(t) = e^{-n^2 t}$

  here $k = -\mu^2 = -n^2$, $c = 1$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} b_n \, e^{-n^2 t} \sin nx$$

- $\boxed{\text{i.c.}}$ $u(x,0) = \sum_{n=1}^{\infty} b_n \sin(nx) = 30 \sin x$

$$\Rightarrow b_1 = 30 \ \& \ b_2 = b_3 = b_4 = \cdots = 0.$$

$$\Rightarrow \boxed{u(x,t) = 30\, e^{-t} \sin(x)}$$

PROBLEM 2

Use the formula given in class to solve the boundary value problem for the wave equation with $L = 1$, $c = 1$, $f(x) = \sin \pi x + \frac{1}{2} \sin 3\pi x + 3 \sin 7\pi x$, $g(x) = \sin 2\pi x$.

$$u(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x)\left(b_n \cos(n\pi t) + b_n^* \sin(n\pi t)\right)$$

$b_1 = 1$, $b_3 = \frac{1}{2}$, $b_7 = 3$, all other $b_n = 0$

$b_2^* = \frac{1}{2\pi}$, all other $b_n^* = 0$

$$\Rightarrow u(x,t) = \sin \pi x \cos \pi t + \frac{1}{2\pi} \sin 2\pi x \sin 2\pi t$$

$$+ \frac{1}{2} \sin 3\pi x \cos 3\pi t + 3 \sin 7\pi x \cos 7\pi t$$

PROBLEM 3

The Lapalce equation in the radially symmetric case can be written as

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}(\frac{\partial^2 u}{\partial \theta^2} + \cot(\theta)\frac{\partial u}{\partial \theta}) = 0,$$

in the spherical coordinates (r, ϕ, θ).

Apply the separation of variables technique, $u = R(r)\Theta(\theta)$, and derive ODEs for R and Θ.

$$u(r,\theta) = R(r)\Theta(\theta)$$

$$\Delta u = R''\Theta + \frac{2}{r}R'\Theta + \frac{1}{r^2}(R\Theta'' + \cot\theta\, R\Theta') = 0$$

$$\Rightarrow (R'' + \frac{2}{r}R')\Theta = -\frac{R}{r^2}(\Theta'' + \cot\theta\,\Theta')$$

$$\Rightarrow \frac{R'' + \frac{2}{r}R'}{-\frac{R}{r^2}} = \frac{\Theta'' + \cot\theta\,\Theta'}{\Theta} = k$$

$$\Rightarrow \begin{cases} R'' + \frac{2}{r}R' + k\frac{R}{r^2} = 0 \\ \Theta'' + \cot\theta\,\Theta' - k\Theta = 0 \end{cases}$$