PROBLEM 1

(a) Derive the *first two* of the (general) orthogonality equations $m, n \geqslant 0$

$$① \int_{-T/2}^{T/2} \cos(\frac{m2\pi}{T}x)\cos(\frac{n2\pi}{T}x)dx = \begin{cases} 0 & \text{if } m \neq n \\ T/2 & \text{if } m = n \end{cases},$$

$$② \int_{-T/2}^{T/2} \cos(\frac{m2\pi}{T}x)\sin(\frac{n2\pi}{T}x)dx = 0.$$

$$\int_{-T/2}^{T/2} \sin(\frac{m2\pi}{T}x)\sin(\frac{n2\pi}{T}x)dx = \begin{cases} 0 & \text{if } m \neq n \\ T/2 & \text{if } m = n \end{cases}.$$

You may use the following identities:

$$\cos a \cos b = \frac{1}{2}[\cos(a-b) + \cos(a+b)], \quad \sin a \cos b = \frac{1}{2}[\sin(a-b) + \sin(a+b)].$$

(b) Assuming $f(x)$ is a piecewise-smooth and continuous T-periodic function, we have the Fourier series (in the real form)

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(\frac{n2\pi}{T}x) + b_n \sin(\frac{n2\pi}{T}x) \right).$$

Use (a) to show that for $n \geq 1$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\cos(\frac{n2\pi}{T}x)dx \quad b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\sin(\frac{n2\pi}{T}x)dx$$

(a) $① = \int_{-T/2}^{T/2} \frac{1}{2}\left[\cos\frac{(m-n)2\pi}{T}x + \cos\frac{(m+n)2\pi}{T}x\right]dx$

$\boxed{\text{if } (m \neq n)} = \frac{1}{2}\left[\frac{T}{(m-n)2\pi}\sin\frac{(m-n)2\pi}{T}x + \frac{T}{(m+n)2\pi}\sin\frac{(m+n)2\pi}{T}x\right]_{-T/2}^{T/2} = 0$

Since $\sin(\frac{N2\pi}{T}x)$ is T-periodic

if $(m = n)$ $① = \int_{-T/2}^{T/2}\left(\frac{1}{2}\cos 0 + \cos\frac{(m+n)2\pi}{T}x\right)dx$

same argument as before

$= \frac{1}{2}\left[\frac{T}{2} - (-\frac{T}{2})\right] + 0 = \frac{T}{2}$

② $\int_{-T/2}^{T/2} \cos(\frac{m2\pi}{T}x)\sin(\frac{n2\pi}{T}x) = 0$

even   odd

(b) $$\frac{2}{T}\int_{-T/2}^{T/2} f(x)\cos\left(\frac{n2\pi}{T}x\right)dx$$

$$= \frac{2}{T}\int_{-T/2}^{T/2}\left[a_0 + \sum_{m=1}^{\infty} a_m\cos\left(\frac{m2\pi}{T}x\right) + b_m\sin\left(\frac{m2\pi}{T}x\right)\right]\underline{\cos\left(\frac{n2\pi}{T}x\right)dx}$$

$$\underset{\text{by (a)}}{=} \frac{2}{T}\, a_n\left(\frac{T}{2}\right) = a_n$$

$$\frac{2}{T}\int_{-T/2}^{T/2} f(x)\sin\left(\frac{n2\pi}{T}x\right)dx$$

$$= \frac{2}{T}\int_{-T/2}^{T/2}\left[a_0 + \sum_{m=1}^{\infty} a_m\cos\frac{m2\pi}{T}x + b_m\sin\left(\frac{m2\pi}{T}x\right)\right]\sin\left(\frac{n2\pi}{T}x\right)dx$$

$$\underset{\text{by (a)}}{=} \frac{2}{T}\, b_n\left(\frac{T}{2}\right) = b_n$$

PROBLEM 2

(a) Let $f(x) = |\sin x|$. What is the smallest period T of $f(x)$?
(b) Find the (real form of) Fourier series of $f(x)$.

§2.2 Exc #7

(a) $T = \pi$.

(b) $f(x)$ is even $\Rightarrow b_n = 0$

$$f(x) = a_0 + \sum_{n=1}^{\infty}\Big(a_n \cos \underbrace{\frac{n2\pi}{T}}_{= 2n} x + b_n \sin \underbrace{\frac{n2\pi}{T}}_{= 2n} x\Big)$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos(2n x)$$

$\boxed{n \geq 1}$

$$a_n = \frac{1}{\pi/2}\int_{0}^{\pi} f(x)\cos(n2x)\,dx$$

$f(x) = |\sin x| > 0$
$= \sin x \quad 0 \leq x \leq \pi$

$$= \frac{2}{\pi}\int_0^{\pi} \sin x \cos(n2x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\frac{1}{2}\Big[\sin((1-2n)x) + \sin((1+2n)x)\Big]dx$$

$$= \frac{1}{\pi}\left[\frac{-1}{1-2n}\cos((1-2n)x) - \frac{1}{1+2n}\cos((1+2n)x)\right]\Big|_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-1}{1-2n}(-1)^{1-2n} - \frac{1}{(1+2n)}(-1)^{1+2n} + \frac{1}{1-2n} + \frac{1}{(1+2n)}\right] = \frac{4}{\pi(1-4n^2)}$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi}\sin x\,dx = \frac{1}{\pi}\left[-\cos x\right]_0^{\pi} = \frac{2}{\pi}.$$

$$\Rightarrow |\sin x| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n)^2-1}\cos 2nx$$

$\boxed{\text{Given}}$: $a_0 = \frac{1}{T}\int_0^{T} f(x)\,dx$

PROBLEM 3

(a) Find the complex form of the Fourier series of the given 2π-periodic function $f(x) = e^{-i2x}$ if $-\pi < x < \pi$. (Hint: you might not need any calculation in part (a).)

(b) Find the complex form of the Fourier series of the given 2π-periodic function e^{ax} if $-\pi < x < \pi$. You may use the fact the (complex) Fourier coefficients are

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(x)e^{-i\frac{n2\pi}{T}x}.$$

(c) Use (b) to find the complex ~~form of the Fourier~~ series of the given 2π-periodic function $f(x) = \cosh 2x$ if $-\pi < x < \pi$. $\boxed{\S 2.6\ \#1}$

(a) $f(x) = e^{-i2x}$ is the Fourier series, i.e. $c_n = \begin{cases} 1 & n = -2 \\ 0 & n \neq -2 \end{cases}$

(b) $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\underbrace{\frac{n2\pi}{T}}_{n}x}$ $(T = 2\pi)$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax}e^{-inx}\,dx = \frac{1}{2\pi}\left[\frac{e^{(a-in)x}}{a-in}\right]_{-\pi}^{\pi} = \frac{(-1)^n}{a-in}\boxed{\frac{\sinh \pi a}{\pi}}$$

$$\frac{\sinh \pi a}{\pi} = \frac{e^{\pi a} - e^{-\pi a}}{2\pi}$$

$$\Rightarrow e^{ax} = \sum_{n=-\infty}^{\infty}\left(\frac{e^{\pi a} - e^{-\pi a}}{2\pi}\right)\left(\frac{(-1)^n}{a-in}\right)e^{inx}$$

(c) $f(x) = \cosh(2x) = \dfrac{e^{2x} + e^{-2x}}{2}$

By (b) $$e^{2x} = \frac{e^{2\pi} - e^{-2\pi}}{2\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{2-in}e^{inx}$$

$$e^{-2x} = \frac{e^{2\pi} - e^{-2\pi}}{2\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{(-2)-in}e^{inx}$$

$$\Rightarrow \cosh(2x) = \left(\frac{1}{2}\right)\frac{e^{2\pi} - e^{-2\pi}}{2\pi}\sum_{n=-\infty}^{\infty}\left[\frac{(-1)^n}{2-in} + \frac{(-1)^n}{-2-in}\right]e^{inx}$$

$$= \frac{e^{2\pi} - e^{-2\pi}}{2\pi}\sum_{n=-\infty}^{\infty}\left[\frac{(-1)^n}{n^2+2^2}\right]e^{inx}$$